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OSCILLATIONS AND ASYMPTOTIC BEHAVIOR OF CERTAIN SECOND ORDER NEUTRAL DIFFERENTIAL EQUATION

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Abstract

The paper deals with the oscillatory and asymptotic behavior of solutions of the second order neutral differential equations with variable coefficients and some generalizations of such kind of equations.

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1. Introduction

The paper deals with the oscillatory and asymptotic behavior of solutions of the second order neutral differential equations of the form

$$(1) \qquad (a(x)(y'(x)+\sum_{i=1}^{k} p_i y'(x-x_i)))' + q(x)f(y(x-x_0)) = 0$$

where $a(x)$, $a'(x)$ and $q(x)$ are continuous functions such that $a(x) > 0$, $q(x) \geq 0$ and not identicly equal to zero in any neighborhood of infinity. p_i are constants and the function f is considered subject to the condition

(2) $\qquad f$ is continuous, nondecreasing and $uf(u) > 0$ for $u \neq 0$.

A real function $g(x)$ has the same property **eventually** if there exists $N \geq 0$ such that $g(x)$ has this property for $x \geq N$.

A nontrivial solution of (1) is said to be **oscillatory** if $y(x)$ is not of the same sign eventually. Otherwise, y is said to be **nonoscillatory**. Equation is called **oscillatory** if every its solution is oscillatory. Otherwise it is called nonoscillatory.

For a neutral differential equation the highest derivative of the unknown function appears with the argument x (the present state of the system) as well as one or more retarded arguments (the past state of the system). Investigations of such equations or systems, beside of their theoretical interest, have some importance in applications (see [2] and [7]).

There is much current interest in oscillation theory of differential equations of neutral type (see [1], [5], [8] and [6]). Not too much is done in the case of continuous coefficients. This is a motivation for our paper.

2. Preliminaries

In what follows we shall use the following lemmas which give a useful information about the bounds for nonoscillatory solutions of the next equation:

$$(a(x)z'(x))' + q(x)f(z(x)) = 0, \; x \geq 0. \tag{3}$$

Lemma 1. *([3]) Consider (3) subject to the conditions (2),*

$$q(x) \geq 0 \text{ and } q(x) \text{ is continuous and not eventually zero,} \tag{4}$$

$a(x)$ is positive and continuous and

$$\int_0^\infty \frac{dx}{a(x)} < \infty. \tag{5}$$

Then, every nonoscillatory solution y of (3) satisfies eventually the following estimate

$$A\rho(x) \leq |y(x)| \leq B$$

for some positive constants A and B (depending on y), where

$$\rho(x) = \int_x^\infty \frac{dt}{a(t)}.$$

Lemma 2. *([4]) Consider (3) subject to conditions (2), (4), $a(x) > 0$ and*

$$\int_0^\infty \frac{dx}{a(x)} = \infty. \tag{6}$$

Then, every nonoscillatory solution y of (3) satisfies eventually the following estimate

$$C \le |y(x)| \le DR(x)$$

for some positive constants C and D (depending on y), where

$$R(x) = \int_0^x \frac{dt}{a(t)}.$$

Near this "a priori" estimates we need the next

Lemma 3. *Suppose that $y(x) > 0$ eventually and define*

$$z(x) = \sum_i^k p_i y(x - x_i),\ p_i > 0. \tag{7}$$

If $p_0 > \sum_{i=1}^k p_i \equiv P$ then $z(x) \to C \ge 0$ if and only if $y(x) \to \frac{C}{P+p_0}$, $x \to \infty$.

Proof. Suppose that $z(x) \to C$, $x \to \infty$ and that

$$\limsup_{x\to\infty} y(x) = \frac{c+q_1}{p_0+P} \text{ and } \liminf_{x\to\infty} y(x) = \frac{c-q_2}{p_0+P}$$

for some $q_1, q_2 \ge 0$. Accordingly to assumptions there are a sequences $\bar{x}_n, \bar{x}_m \to \infty$; $n, m \to \infty$ such that

$$y(\bar{x}_n) \to \frac{c+q_1}{p_0+P}.\quad y(\bar{x}_m) \to \frac{c-q_2}{p_0+P},\ n, m \to \infty.$$

a) Suppose that $q_1 \ge q_2$ and $q_1 > 0$. Taking $x = \bar{x}_n + x_0$. (7) implies that

$$\begin{aligned} C &= \frac{c+q_1}{p_0+P}p_0 + \lim_{n\to\infty}\sum_{i=1}^k p_i y(\bar{x}_n + x_0 - x_i) \\ &= \frac{c+q_1}{p_0+P}p_0 + \sum_{i=1}^k \frac{c+\delta_i}{p_0+P}p_i, \end{aligned}$$

where $-q_2-\varepsilon<\delta_i<q_1+\varepsilon$ eventually for every $\varepsilon>0$. The above equation implies

$$p_0q_1=-\sum_{i=1}^{k}\delta_ip_i<\sum_{i=1}^{k}p_i(q_2+\varepsilon).$$

Choosing $\varepsilon<\frac{p_0-P}{P}q_1$ we get $q_1<q_2$, a contradiction.
b) Suppose that $q_2\geq q_1$ and $q_2>0$. Taking $x=\bar{x}_m+x_0$, (7) implies that

$$\begin{aligned} C &= \frac{c-q_2}{p_0+P}p_0+\lim_{m\to\infty}\sum_{i=1}^{k}p_iy(\bar{x}_m+x_0-x_i) \\ &= \frac{c-q_2}{p_0+P}p_0+\sum_{i=1}^{k}\frac{c+\delta_i}{p_0+P}p_i, \end{aligned}$$

where $-q_2-\varepsilon<\delta_i<q_1+\varepsilon$ eventually, for every $\varepsilon>0$. The above equation implies

$$p_0q_2=\sum_{i=1}^{k}\delta_ip_i<\sum_{i=1}^{k}p_i(q_1+\varepsilon).$$

Choosing $\varepsilon<\frac{p_0-P}{P}q_2$ we get $q_2<q_1$, a contradiction.

As the convergence of $y(x)$ implies the convergence of $y(x-x_i)$ to the same limit, the proof of the second part of lemma is obvious. □

Remark 1. *If $p_0=P$, $y(x)=2+\sin x$ for appropriate x_i could be a contraexample.*

3. Oscillations and asymptotic behavior

Consider the second order neutral differential equation (1) where we additionally suppose that $P<1$. Then

Theorem 1. *If*

$$\int_{t_0}^{\infty}q(t)dt=\infty \text{ and } \int_{t_0}^{\infty}\frac{dx}{a(x)}\int_{t_0}^{x}q(t)dt=\infty$$

then every solution y of (1) is either oscillatory or else $y(x)\to 0$ as $x\to\infty$.

Proof. Let y be a nonoscillatory solution of (1). Suppose, without loss of generality, that $y(x) > 0$ eventually. This implies that $y(x - x_i) > 0$ eventually. Set

$$z(x) = y(x) + \sum_{i=1}^{k} p_i y(x - x_i). \tag{8}$$

Then $z(x) > y(x) > 0$ eventually.

According to (1) we have $(a(x)z'(x))' < 0$ eventually. Thus, either $z'(x) > 0$ or $z'(x) < 0$ eventually.

a) Assume that $z'(x) > 0$ eventually. It follows that $z'(x - x_0) > 0$ eventually and by (8)

$$y(x - x_0) = z(x - x_0) - \sum_{i=1}^{k} p_i y(x - x_0 - x_i)$$

$$\geq z(x - x_0) - \sum_{i=1}^{k} p_i z(x - x_0 - x_i) \geq z(x - x_0) - \sum_{i=1}^{k} p_i z(x - x_0),$$

what implies

$$y(x - x_0) \geq (1 - P)z(x - x_0). \tag{9}$$

Define a positive function $w(x)$ such that

$$w(x) = \frac{a(x)z'(x)}{f((1 - P)z(x - x_0))}.$$

Then

$$\begin{aligned} w'(x) &= \frac{(a(x)z'(x))'}{f((1 - P)z(x - x_0))} \\ &- \frac{a(x)z'(x)f'((1 - P)z(x - x_0))(1 - P)z'(x - x_0)}{f^2((1 - P)z(x - x_0))} \end{aligned}$$

Condition (2) together with (9) yields to

$$w'(x) \leq -q(x),$$

which, after integration, gives

$$w(x) \leq w(t_0) - \int_{t_0}^{x} q(t)dt.$$

Letting $x \to \infty$ we get immediate contradiction.
b) Assume that $z'(x) < 0$ eventually. Then $\lim_{x\to\infty} z(x) = C$ and suppose that $C > 0$. Accordingly to Lemma 3 $lim_{x\to\infty} y(x-x_0) = \frac{c}{1+P}$ which implies that $y(x-x_0) \geq \frac{c}{2(1+P)}$ eventually. Thus

$$(a(x)z'(x))' \leq -f(\frac{c}{2(1+P)})q(x)$$

eventually. Integrating the above inequality from t_0 to x we get

$$a(x)z'(x) \leq a(t_0)z'(t_0) - f(\frac{c}{2(1+P)}) \int_{t_0}^{x} q(t)dt.$$

It yields to

$$a(x)z'(x) \leq -f(\frac{c}{2(1+P)}) \int_{t_0}^{x} q(t)dt.$$

Dividing by $a(x)$ and integrating from t_0 to x we get

$$z(x) \leq z(t_0) - f(\frac{c}{2(1+P)}) \int_{t_0}^{x} \frac{dt}{a(t)} \int_{t_0}^{x} q(s)ds.$$

Letting $x \to \infty$, the right side of the last inequality tends to $-\infty$ and it is a contradiction to the fact that $z(x) \to C > 0$, and the proof is complete. □

Next theorem guarantees that all solutions of (1) are oscillatory.

Theorem 2. *If*

$$p < 1, \int_{t_0}^{\infty} q(t)dt = \infty \text{ and } \int_{t_0}^{\infty} a(t)dt = \infty$$

then the equation (1) is oscillatory.

Proof. Let $y(x)$ be a nonoscillatory solution of (1). Without loss of generality we may suppose that $y(x) > 0$ eventually. As in the proof of Theorem 1, the third condition of theorem, as it was shown in [4], implies that $z'(x) > 0$ eventually. As the proof follows the same line as in the case a) of Theorem 1 it will be omitted.

Question is what happens when $\int_{t_0}^{dt} a(t)$ converges. The answer gives the next

Theorem 3. *If* $p < 1$, $\int_{t_0}^{dt} a(t) < \infty$ *and* $\int_{t_0}^{x} \frac{dt}{a(t)} \int_{t_0}^{x} q(s)ds = \infty$. *Then every solution* y *of (1) is either oscillatory or else* $y(x) \to 0$ *as* $x \to \infty$.

Proof. As in the proof of Theorem 1 we differ two cases
a) Assume that $z'(x) > 0$ eventually. Accordingly to Lemma 1 we have that $\lim_{x\to\infty} z(x) = c > 0$ which by Lemma 3 gives that $\lim_{x\to\infty} y(x) = \frac{c}{1+P}$ and the estimate $y(x - x_0) > \frac{z(x-x_0)}{2(1+P)}$ eventually. Conditions $\int_{t_0}^{\infty} \frac{dt}{a(t)} < \infty$ and $\int_{t_0}^{\infty} \frac{dt}{a(t)} \int_{t_0}^{t} q(s)ds = \infty$ imply that $\int_{t_0}^{\infty} q(t)dt = \infty$ and we can proceed as in the proof of the case a) of Theorem 1.
b) Assume that $z'(x) < 0$ eventually. According to the observations given in the case a) the proof follows the same line as the proof of the case b) in Theorem 1. □

Remark 2. *In the light of Lemma 3 we are able to generalize assertions of our theorems for the differential equation (3) where z is given by (7).*

Theorems 1 and 2 are improvement and generalizations of Theorem 1 and 2 in [5].

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REZIME

OSCILACIJE I ASIMPTOTSKO PONAŠANJE NEKIH NEUTRALNIH DIFERENCIJALNIH JEDNAČINA

U radu se bavimo oscilatornim i asimptotskim ponašanjem rešenja neutralnih diferencijalnih jednačina drugog reda sa promenljivim koeficijentima i nekim generalizacijama takvih jednačina.

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