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NEW GENERALIZED FUNCTIONS OF $\mathcal{K}'\{M_p\}$ TYPE

Arpad Takači
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

In this note we introduce and analyze an analog of new tempered generalized functions (see [1], part II), which corresponds to the spaces of $\mathcal{K}'\{M_p\}$ type from [2]. In particular, we prove that if the sequence of functions $(M_p)_{p\in\mathbf{N}_0}$ meets some usual conditions, then the introduced space allows an inner multiplication.

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1. Introduction

The spaces of $\mathcal{K}'\{M_p\}$ type were introduced in the fifties by Gel'fand and Shilov (see [2]). Their approach gave a new view and deeper understanding of (up to then used) most important subspaces of distributions, and at the same time, enabled the introduction of some other spaces, which in general, cannot be imbedded in the space of distributions $\mathcal{D}'$. Such examples can be found, for instance, in [3], [7] or [5]. Moreover, several spaces defined by A. H. Zemanian and his collaborators are of this kind (see [6]).

Some deeper properties of $\mathcal{K}'\{M_p\}$ spaces were analyzed in many papers. For instance, in [4], the conditions for the existence of the convolution in

$\mathcal{K}'\{M_p\}$ spaces were given via the properties of the sequence of functions $(M_p)_{p\in\mathbf{N}_0}$ type. However, the problem of multiplication in $\mathcal{K}'\{M_p\}$ spaces was not so widely observed. This is not strange, since, as proved long time ago by L. Schwartz, the multiplication of distributions can not be solved in a "satisfactory" way. In fact, he proved that there is no associative algebra containing the space of distributions, which would retain the multiplication of continuous functions with its good properties, like commutativity and associativity.

In his book [1], J. F. Colombeau introduced the space of new generalized functions, denoted by $\mathcal{G}[\mathbf{R}^n]$, in which the introduced multiplication turned out to be an inner operation. Colombeau's multiplication in $\mathcal{G}[\mathbf{R}^n]$ exactly generalizes the multiplication of infinitely differentiable functions, but, as could have been suspected, not that of continuous functions. Let us remark here that this "defect" was repaired with the notion of "association" between elements from $\mathcal{G}[\mathbf{R}^n]$ (see [1], p. 64).

In the "classical" case of distributions, the space of tempered distributions played a central role when the Fourier transformation was to be introduced. In the case of new generalized functions, their analogue were the "tempered new generalized functions ", introduced in [1] and denoted there by $\mathcal{G}_\tau[\mathbf{R}^n]$.

In this paper, in a similar manner we introduce the "$\mathcal{K}'\{M_p\}$ new generalized functions". Firstly, we give an outline of $\mathcal{K}'\{M_p\}$ spaces and some elements of the theory of new generalized functions. Secondly, we define the announced space $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$, and compare it to the spaces $\mathcal{G}[\mathbf{R}^n]$, and $\mathcal{K}'\{M_p\}$. Finally, assuming a condition from [2], we show that it is an algebra with the multiplication defined as in [1].

2. Preliminaries

Throughout the paper we shall assume that $\alpha = (\alpha_1, .., \alpha_n) \in \mathbf{N_0}^n$ is a multiindex and $|\alpha| := \alpha_1 + .. + \alpha_n$. As usual, $\partial^\alpha f(x)$ is the partial derivative of order $|\alpha|$, in either classical or generalized sense, depending on the context. Further on, D will stand for any linear differential operator with constant coefficients (shortly: a derivation operator).

2.1. The space $\mathcal{K}'\{M_p\}$

Let $\mathcal{M} := (M_p)_{p\in\mathbf{N}_0}$ be a sequence of continuous functions

$$M_p : \mathbf{R}^n \to [1,\infty),$$

such that

$$1 = M_0(x) \le M_1(x) \le M_2(x) \le \dots, \quad x \in \mathbf{R}^n.$$

Additionally, we shall suppose that the following three conditions on $\mathcal{M}$ from [2] hold:

(M) for every $p \in \mathbf{N}$ there exists a constant $C_{p,i}$ such that

$$|x_i'| \le |x_i''| \Rightarrow M_p(x_1,..,x_i',..,x_n) \le C_{p,i} M_p(x_1,..,x_i'',..,x_n);$$

(P) for every ε and every $p \in \mathbf{N}$ there exists a natural number $p' > p$, such that for some sufficiently large K it holds

$$\| x \| > K \Rightarrow M_p(x) \le \varepsilon M_{p'}(x);$$

(N) for every $p \in \mathbf{N}$ there exists a natural number $p' > p$, such that the function

$$m_{pp'}(x) := \frac{M_p(x)}{M_{p'}(x)}$$

tends to zero as $\| x \| \to \infty$ and it is an integrable function on $\mathbf{R}^n$.

By definition, $\mathcal{K}\{M_p\}$ denotes the locally convex space of infinitely differentiable functions on $\mathbf{R}^n$, endowed with the family of seminorms $(\gamma_p)_{p\in\mathbf{N}_0}$, where

$$\gamma_p(\phi) := \sup\{M_p(x)\,|\partial^\alpha\phi(x)|,\ x \in \mathbf{R}^n, |\alpha| \le p\},$$

for an infinitely differentiable function ϕ. Clearly, such a ϕ is in $\mathcal{K}\{M_p\}$ iff for every $p \in \mathbf{N}$ it holds $\gamma_p(\phi) < \infty$. (In the terminology of [6], $\mathcal{K}\{M_p\}$ is a countable multinormed space.)

The space of continuous linear functionals on $\mathcal{K}\{M_p\}$ is denoted by $\mathcal{K}'\{M_p\}$, and we shall endow it with the weak topology. In order to obtain convenient representation for the elements of $\mathcal{K}'\{M_p\}$, we shall use a condition from [4]:

(N') for every $p \in \mathbf{N}$ and every $i \in \{1, 2, .., n\}$ there exist a natural number $p_i > p$ and a constant $B_{p,i} > 0$ such that the function

$$\sup\{\int_{\mathbf{R}} \frac{M_p(x)}{M_{p_i}(x)} dx_i, \ \ \overline{x_i} := (x_1, .., x_{i-1}, x_{i+1}, .., x_n) \in \mathbf{R}^{n-1}\} \leq B_{p,i}.$$

In fact, the condition (N) follows from (N'), while (N)=(N') if $n = 1$. It was proved in [4] that if the conditions (M), (P) and (N') hold, then a linear functional T on $\mathcal{K}\{M_p\}$ is also continuous iff there exist a multiindex $\alpha \in \mathbf{N_0}^n$ and a bounded continuous function f on $\mathbf{R}^n$ such that

$$T = \partial^\alpha (f(x) M_{|\alpha|}(x)) \tag{1}$$

in the sense of $\mathcal{K}'\{M_p\}$.

Probably the most important case of the sequence $\mathcal{M}$ one gets if the functions M_p are given by

$$M_p(\| x \|) = (1 + \| x \|)^{p/2}, \ p \in \mathbf{N_0}. \tag{2}$$

Then, the space $\mathcal{K}\{M_p\}$ and its dual $\mathcal{K}'\{M_p\}$ become the space of rapidly decreasing functions $\mathcal{S}(\mathbf{R}^q)$ and the space of tempered distributions $\mathcal{S}'(\mathbf{R}^q)$. From the many other cases of interest, let us mention here the so called exponential distributions analyzed, for instance, in [3], [7] and [5].

2.2. New generalized functions

Colombeau's new generalized functions are the elements of the space $\mathcal{G}[\mathbf{R}^n]$, obtained as the equivalence classes in the space $\mathcal{E}_M[\mathbf{R}^n]$, modulo the space $\mathcal{N}[\mathbf{R}^n]$. In order to define these spaces, J. F. Colombeau defined the sets A_p, $p \in \mathbf{N}$, as follows:

$$A_p = \{\phi \in \mathcal{D} \mid \int_{\mathbf{R}^n} \phi(x)\, dx = 1, \ \int_{\mathbf{R}^n} x^\alpha \phi(x)\, dx = 0, \ 1 \leq |\alpha| \leq p\}.$$

For $\varepsilon > 0$ and a function ϕ on $\mathbf{R}^n$, put $\phi_\varepsilon(x) := \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$. Then let

$$\mathcal{E}_M[\mathbf{R}^n] = \{R : A_1 \times \mathbf{R}^n \to \mathbf{C} \mid \ (\forall \phi \in A_1) \ R(\phi, \cdot) \in C^\infty(\mathbf{R}^n),$$

$$(\forall K, \ K \text{ compact in } \mathbf{R}^n) \ (\forall D, \ D \text{ a derivation operator })$$

$$(\exists N \in \mathbf{N}) \text{ such that } \ (\forall \phi \in A_1) \ (\exists c, \eta > 0):$$

$$|DR(\phi_\varepsilon, x)| \leq c \cdot \varepsilon^{-N}, \quad x \in K,\ 0 < \varepsilon < \eta\}.$$

Let now Γ denote the set of monotonically increasing sequences of positive numbers diverging to infinity. Then put

$$\mathcal{N}[\mathbf{R}^n] = \{R \in \mathcal{E}_M[\mathbf{R}^n] \mid \ (\forall K,\ K \text{ compact in } \mathbf{R}^n)$$
$$(\forall D,\ D \text{ a derivation operator })\ (\exists N \in \mathbf{N})\ (\exists \alpha \in \Gamma)$$
$$\text{such that } \forall \phi \in A_q,\ q \geq N,\ \ (\exists c, \eta > 0):$$
$$|DR(\phi_\varepsilon, x)| \leq c \cdot \varepsilon^{\alpha(q) - N}, \quad x \in K,\ 0 < \varepsilon < \eta\}.$$

One can easily check that $\mathcal{E}_M[\mathbf{R}^n]$ is an algebra (with the usual multiplication of functions), while $\mathcal{N}[\mathbf{R}^n]$ is an ideal in it. The equivalence relation in $\mathcal{E}_M[\mathbf{R}^n]$ defined by

$$R_1 \sim R_2 \Longleftrightarrow R_1 - R_2 \in \mathcal{N}[\mathbf{R}^n],$$

leads to the space of equivalence classes $\mathcal{G}[\mathbf{R}^n]$. In other words,

$$\mathcal{G}[\mathbf{R}^n] = \mathcal{E}_M[\mathbf{R}^n]/\mathcal{N}[\mathbf{R}^n].$$

This space, whose elements will be called "new generalized functions", is an algebra if the multiplication of two elements, say G_1 and G_2, is defined as the class of the product $R_1 \cdot R_2$, where R_i, $i = 1, 2$, is any representative of G_i. This multiplication is coherent with that of infinitely differentiable functions; however, it is not coherent with the multiplication of just continuous ones. In fact, $C([\mathbf{R}^n])$ is not a subalgebra of $\mathcal{G}[\mathbf{R}^n]$. Furthermore, if D is a derivation operator, then the derivative of $G \in \mathcal{G}[\mathbf{R}^n]$ is the class of DR, where R is any representative of G. One easily checks that D maps $\mathcal{G}[\mathbf{R}^n]$ into itself and that the Leibniz formula holds.

Here we have to add some notions from the so–called generalized analysis, in fact, to rewrite the definition from [1] of the algebra of generalized complex numbers $\overline{\mathbf{C}}$. Just like the space of new generalized functions, the last is obtained as the set of equivalence classes of the elements from the set $\mathcal{E}_M$ over its ideal $\mathcal{I}$. Here

$$\mathcal{E}_M = \{R : A_1 \to \mathbf{C} \mid \ \text{such that } (\exists N \in \mathbf{N}) \text{ such that } (\forall \phi \in A_N)(\exists c, \eta > 0):$$
$$|R(\phi_\varepsilon)| \leq c \cdot \varepsilon^{-N}, \quad 0 < \varepsilon < \eta\},$$

and

$$\mathcal{I} = \{R : A_1 \to \mathbf{C} \mid \ \text{such that } (\exists N \in \mathbf{N})\ (\exists \alpha \in \Gamma)$$

$$\text{such that } (\forall q \geq N)\ (\forall \phi \in A_q)\ (\exists c, \eta > 0):$$
$$|R(\phi_\varepsilon)| \leq c \cdot \varepsilon^{\alpha(q)-N}, \quad 0 < \varepsilon < \eta\}.$$

Then

$$\overline{\mathbf{C}} = \mathcal{E}_M/\mathcal{I}.$$

Of course, the set of complex numbers $\mathbf{C}$ can be injectively imbedded into $\overline{\mathbf{C}}$ via the maps

$$R_z(\phi) := z$$

for every $z \in \mathbf{C}$.

3. $\mathcal{K}'\{M_p\}$ new generalized functions

In [1], Chapters 4-6, a theory of "tempered new generalized functions" was given; as can be suspected, they correspond to tempered distributions. In an analogous way we introduce the "$\mathcal{K}'\{M_p\}$ new generalized functions", which as we shall see, will correspond to to the elements of the $\mathcal{K}'\{M_p\}$ space. To that end, let us start with the spaces $\mathcal{E}_{(M_p)}[\mathbf{R}^n]$ and $\mathcal{N}_{(M_p)}[\mathbf{R}^n]$:

$$\mathcal{E}_{(M_p)}[\mathbf{R}^n] = \{R : A_1 \times \mathbf{R}^n \to \mathbf{C} \mid \ (\forall \phi \in A_1)\,(R(\phi.\cdot) \in C^\infty(\mathbf{R}^n),$$

$$(\forall D,\ D \text{ a derivation operator })\,(\exists p \in \mathbf{N}) \text{ such that } (\forall \phi \in A_p)\,(\exists c, \eta > 0):$$
$$|DR(\phi_\varepsilon, x)| \leq c \cdot M_p(x) \cdot \varepsilon^{-p},\ \ x \in \mathbf{R}^n.\ 0 < \varepsilon < \eta\}\};$$

$$\mathcal{N}_{(M_p)}[\mathbf{R}^n] = \{R \in \mathcal{E}_{M_p}[\mathbf{R}^n] \mid \ (\forall D,\ D \text{ a derivation operator })$$
$$(\exists p \in \mathbf{N})\,(\exists \alpha \in \Gamma) \quad \text{such that if } p' > p \text{ then } (\forall \phi \in A_{p'})\,(\exists c, \eta > 0):$$
$$|DR(\phi_\varepsilon, x)| \leq c \cdot M_p(x) \cdot \varepsilon^{\alpha(p')-p},\ \ x \in \mathbf{R}^n,\ 0 < \varepsilon < \eta\}.$$

Since we want $\mathcal{E}_{(M_p)}[\mathbf{R}^n]$ to be an algebra, we suppose additionally that the sequence $(M_p)_{p\in\mathbf{N}}$ satisfies the following condition, denoted by (A) (compare to [2], p.102, rel. (3)).

(A) For every pair $(p_1, p_2) \in \mathbf{N}^2$ there exists an integer $p_3 > max(p_1, p_2)$ and a constant $C_{p_1,p_2} > 0$ such that

$$M_{p_1}(x) \cdot M_{p_2}(x) \leq C_{p_1,p_2} \cdot M_{p_3}(x) \text{ for } |x| > p_3.$$

Clearly this condition holds for tempered distributions (see (2)), but also in the space of exponential distributions, obtained by the sequence

$$M_p(x) = \exp(p \cdot |x|^k), \quad p \in \mathbf{N_0}, \quad \text{for fixed } k \in \mathbf{N}.$$

The condition (A) easily implies that $\mathcal{N}_{(M_p)}[\mathbf{R}^n]$ is an ideal in $\mathcal{E}_{(M_p)}[\mathbf{R}^n]$. So, as in the case of new generalized functions, we can introduce the following equivalence relation in $\mathcal{E}_{(M_p)}[\mathbf{R}^n]$:

$$R_1 \sim R_2 \iff R_1 - R_2 \in \mathcal{N}_{(M_p)}[\mathbf{R}^n].$$

The corresponding set of equivalence classes we denote by $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$, i.e.,

$$\mathcal{G}_{(M_p)}[\mathbf{R}^n] = \mathcal{E}_{(M_p)}[\mathbf{R}^n]/\mathcal{E}_{(M_p)}[\mathbf{R}^n]$$

and its elements we call "new generalized functions of the $\mathcal{K}'\{M_p\}$ type". Again, in view of the condition (A), $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$ is an algebra, with the multiplication defined as in $\mathcal{G}[\mathbf{R}^n]$. Moreover, in view of (M) and (P), the derivation is an inner operation in $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$, and the Leibniz formula holds. Of course, the derivative DG of a new generalized function G of the $\mathcal{K}'\{M_p\}$ type is defined as the class of DR, where R is any representative of G.

We shall analyze next the relation between the space $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$ and the whole space $\mathcal{G}[\mathbf{R}^n]$. A canonical map $\mathbf{M}$ from $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$ into $\mathcal{G}[\mathbf{R}^n]$ is given by:

$$\mathbf{M}(R + \mathcal{N}_{(M_p)}[\mathbf{R}^n]) := R + \mathcal{N}_{(M_p)}[\mathbf{R}^n]. \tag{3}$$

Analogously to Proposition 4.1.6 and Remark 4.1.9 in [1], it holds

Theorem 1. *The map from (3) is neither injective nor surjective.*

Proof. The proof of the non–surjectivity of M is the same as in [1]. For the non–injectivity, we shall only outline the necessary modifications for our case.

The goal is to construct an element from the intersection of $\mathcal{E}_{(M_p)}[\mathbf{R}^n]$ and $\mathcal{N}[\mathbf{R}^n]$, which, however, does not belong to $\mathcal{N}_{(M_p)}[\mathbf{R}^n]$. To that end, put first

$$M(x) := \begin{cases} M_0(x), & 0 \le x < 1; \\ M_p(x), & p \le x < p+1,\ p \in \mathbf{N} \end{cases}$$

which we evenly continue to the whole $\mathbf{R}$. Let then ω be an infinitely differentiable non - negative function on $\mathbf{R}$ such that

$$\operatorname{supp}\omega \subset [0,1], \quad \int_{\mathbf{R}} \omega(x)\,dx = 1.$$

The function N defined as the convolution of the functions M and ω

$$N(x) := M * \omega(x) = \int_{\mathbf{R}} M(t)\omega(x-t)\,dt, \quad x \in \mathbf{R},$$

is then infinitely differentiable on $\mathbf{R}$, but does not belong to $\mathcal{K}'\{M_p,$ (More precisely, it does not define a regular element of $\mathcal{K}'\{M_p\}$.)

Let us define now a mapping $C_0 : \mathcal{A}_\infty \to \mathbf{C}$ by

$$C_0(\phi) = \int_{\mathbf{R}} (e^{N(x)} - e^{N(0)})\phi(x)\,dx.$$

It is easy to show that C_0 belongs to $\mathcal{I}$; we omit the details. Then we put for $\phi \in A_1$ whose support's diameter is not greater than 1:

$$\gamma_n(\phi_\varepsilon) = \varepsilon^{-n-1} C_0(\phi_\varepsilon), \quad n \in \mathbf{N}.$$

Putting finally

$$C_n(\phi_\varepsilon) := \begin{cases} \gamma_n(\phi_\varepsilon) & \text{for } \gamma_n(\phi_\varepsilon) \le 1; \\ 1 & \text{for } \gamma_n(\phi_\varepsilon) > 1, \end{cases}$$

we obtain a sequence from $\mathcal{I}$ such that

$$|C_n(\phi)| \le 1 \quad \text{for every } \phi \in A_1.$$

Moreover, for every $n \in \mathbf{N}$ there exists a ϕ from A_n such that $C_n(\phi)$ does not tend to 0 as $\varepsilon \to 0+$.

Put now for $\phi \in A_1$

$$R(\phi_\epsilon) = \sum_{n=1}^{\infty} C_n(\phi_\varepsilon)\omega(x-n). \tag{4}$$

Then clearly $R \in \mathcal{E}_{(M_p)}$ and $R \in \mathcal{N}[\mathbf{R}^n]$. However, we shall show that $R \notin \mathcal{E}_{(M_p)}$. Namely, if the last statement were false, then there would exist a natural number p and a sequence $\alpha \in \Gamma$ such that when $q \geq p$ and $\phi \in A_q$, then there exist positive numbers c and η such that

$$|R(\phi_\epsilon, x)| \leq c \cdot M_p(x)\varepsilon^{\alpha(q)-p}, \quad x \in \mathbf{R},\ 0 < \varepsilon < \eta.$$

Take now $x_0 \in [0,1]$ such that $\omega(x_0) \neq 0$. Then

$$|R(\phi_\epsilon, x_0 + n)| \leq c \cdot M_p(x_0 + n)\varepsilon^{\alpha(q)-p},$$

which means that $R(\phi_{\epsilon, x_0+n})$ tends to 0 as $n \to \infty$. This is contradiction with the definition of R from (4), once we put there $n = q$ □

If $\mathcal{G}_c[\mathbf{R}^n]$ denotes the subspace of $\mathcal{G}[\mathbf{R}^n]$ consisting of new generalized functions with compact support, then it holds:

Theorem 2. *The map* $\mathbf{M_1}$ *from* $\mathcal{G}_c[\mathbf{R}^n]$ *into* $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$ *given by*

$$\mathbf{M_1}(G) := \mathbf{M_1}(R + \mathcal{N}[\mathbf{R}^n]) = R + \mathcal{N}_{(M_p)}[\mathbf{R}^n]$$

is injective.

Proof. If a new generalized function G has compact support, then by definition it has a representative R in $\mathcal{G}[\mathbf{R}^n]$ such that if ρ is an infinitely differentiable function on $\mathbf{R}$ with compact support, then $\mathbf{M_1}(G)$ depends neither on R nor on the function ρ. Since $\rho R - R \in \mathcal{N}[\mathbf{R}^n]$, it follows that that the map

$$\mathbf{MM_1} : \mathcal{G}_c[\mathbf{R}^n] \to \mathcal{G}[\mathbf{R}^n]$$

is injective. Hence, $\mathbf{M_1}$ is also injective. □

We relate now the space $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$ with the space $\mathcal{K}'\{M_p\}$. As before, we suppose that the conditions (M), (P), (A) and, additionally, the condition (N') hold. The representation (1) suggests the following definition.

Definition 1. *A new generalized function T from $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$ is a $\mathcal{K}'_1\{M_p\}$ distribution if there exists a continuous function f on $\mathbf{R}^n$ such that*

$$|f(x)| \leq C \cdot M_p(x), \quad x \in \mathbf{R}^n, \tag{5}$$

for some $C > 0$ and $p \in \mathbf{N_0}$ with the property $T = \partial^\alpha f$ for some multiidnex α, in the sense of new generalized functions.

It is clear that $\mathcal{K}'_1\{M_p\}$ is a subset of $\mathcal{G}_{(M_p)}[\mathbf{R}^n]$ and contains the space $\mathcal{E}'(\mathbf{R}^n)$ of distributions with compact support. In view of the representation (1), the sets $\mathcal{K}'\{M_p\}$ and $\mathcal{K}'_1\{M_p\}$ are equal, so it is convenient to denote the latter also by $\mathcal{K}'\{M_p\}$. Our goal is to prove that though the mapping $\mathbf{M}$ from (3) is not injective, its restriction to $\mathcal{K}'\{M_p\}$ still is. For that goal, we need the forthcoming theorem, which might be of some self interest, too. In the statement and its proof we use the so-called generalized integrals; this notion was developed in [1], Chapter 2.

Theorem 3. *If $T \in \mathcal{K}'\{M_p\}$ and $\theta \in \mathcal{K}\{M_p\}$, then the generalized integral*

$$\int_{\mathbf{R}^n} T(x)\,\theta(x)\,dx$$

is a (classical) complex number.

Proof. Let f, α and p be as in (5) from Definition 1. Then in $\overline{\mathbf{C}}$ (the set of generalized complex numbers), we have

$$\int_{\mathbf{R}^n} T(x)\,\theta(x)\,dx = (-1)^{|\alpha|} \int_{\mathbf{R}^n} f(x)\,\partial^\alpha \theta(x)\,dx,$$

where $|\alpha|$ is the order of the operator ∂^α. The last integral is in the class of

$$R_1 : \phi_\varepsilon \mapsto (-1)^{|\alpha|} \int_{\mathbf{R}^{2n}} f(x+\varepsilon y)\,\partial^\alpha \theta(x)\,\phi(\varepsilon x)\,\phi(y)\,dx, \phi \in \mathcal{D}.$$

Then, as in Theorem 4.2.2 from [1], one proves that the map $R_2 : A_1 \to \overline{\mathbf{C}}$ defined by

$$R_2 : \phi_\varepsilon \mapsto (-1)^{|\alpha|} \int_{\mathbf{R}^n} f(x)\,\partial^\alpha \theta(x)\,\phi(\varepsilon x)\,dx, \phi \in \mathcal{D},$$

is also in the same class. Namely, let p' be the integer corresponding to p, whose existence follows from (N'). Then, we use the following inequality

$$\left| \int_{\mathbf{R}^n} f(x)\,\partial^\alpha \theta(x)\,dx \right| \le C' \cdot \int_{\mathbf{R}^n} M_p\,M_{p'}^{-1}(x)\,dx. \tag{6}$$

Now the integral on the left hand side of (6) converges, in the classical sense. This means that the generalized integral on the left hand side and the classical one on the right hand side of the following relation are equal:

$$\int_{\mathbf{R}^n} T(x)\,\theta(x)\,dx = (-1)^{|\alpha|} \int_{\mathbf{R}^n} f(x)\,\partial^\alpha \theta(x)\,dx, \tag{7}$$

and this is what we had to prove. □

Using the representation of continuous functions in $\mathcal{G}[\mathbf{R}^n]$, from relation (7) it follows that the map

$$R : A_1 \times \mathbf{R}^n \to \mathbf{C}$$

given by

$$R(\phi, x) = \int_{\mathbf{R}^n} T(y)\,\phi(y-x)\,dy \tag{8}$$

is a representative of T. Finally, we have come to

Theorem 4. *The restriction of the map* $\mathbf{M}$, *given by (3), to the space* $\mathcal{K}'\{M_p\}$ *is injective.*

Proof. If $T \in \mathcal{K}'\{M_p\}$ satisfies $\mathbf{M}(T) = 0$, then the integral (7) is zero for every $\theta \in \mathcal{D}$. But in view of Theorem 3, it follows that the mapping $R(\cdot,\cdot)$ from (8) is zero. Hence $T = 0$, as claimed.

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REZIME

PROSTORI NOVIH UOPŠTENIH FUNKCIJA $\mathcal{K}'\{M_p\}$ TIPA

U radu se uvodi prostor novih uopštenih (Kolomboovih) funkcija $\mathcal{K}'\{M_p\}$ tipa, koji uz odredjene uslove o nizu $(M_p)_{p\in\mathbf{N}_0}$ čini algebru. Pokazuje se da se prostor tipa Geljfand–Šilov može injektivno smestiti u taj prostor.

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