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CATEGORICAL CHARACTERIZATION OF SOME FAMILIES OF SETS IN A ⊥–DECOMPOSABLE MEASURE SPACE

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Abstract

Using the continuity of t–conorm $\perp$ decomposable measure m with respect to suitably chosen submeasure the completness of the semimetric space $(\mathcal{S}, \Sigma, m)$ is proved. It is proved under some assumptions on a family $\mathcal{S}$ of sets in (S, Σ, m), that either $\mathcal{S} = \Sigma$ or $\mathcal{S}$ is of the first Bair category in (S, Σ, m).

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We have investigated in papers [6] and [7] ⊥–decomposable measures for t–conorm $\perp$ which is contonuous at zero. Such measures are important in probability theory and statistics [9], [10] and in some more general form also in the theory of nonlinear partial differential equations [8]. The starting point for this paper was a topological correspodence of a ⊥–decomposable measure with a submeasure, which result was obtained in [6]. Using this result we prove the completeness of the semimetric space (S, Σ, m). The main results, Theorems 4 and 5, assert that under some assumptions, a family $\mathcal{S}$ of sets in (S, Σ, m) coincides with Σ or is of first category in (S, Σ, m). This generalizes the results of L. Drewnowski [3] obtained for a finite measure space (S, Σ, λ) with respect to the usual FN (Frechet - Nikodym)–topology [4, III.7.1]. The starting point for Drewnowski's result was the result of R. Anantharaman [1]: If (S, Σ, λ) is the usual Lebesgue measure space on $S = [0,1]$ and $F : \Sigma \to l_2$ is the vector measure defined by $F(E) = 2(\int_E r_n d\lambda)$, where (r_n) is the Rademacher sequence, then $F^{-1}(l_p)$ for $1 \leq p < 2$ is of the first category in (S, Σ, λ) equipped with FN–semimetric $(A, B) \to \lambda(A \Delta B)$.

We begin by fixing some notations and terminology (as in [6] and [7]).

Definition 1. *A function* $\perp : [0,1] \times [0,1] \to [0,1]$ *will be called a* t*–conorm if it satisfies*

(A) $\perp(x, 0) = \perp(0, x) = x \quad (x \in [0,1])$,

(B) if $x_1 \leq x_3$ *and* $x_2 \leq x_4$ *for* $x_1, x_2, x_3, x_4 \in [0,1]$ *then* $\perp(x_1, x_2) \leq \perp(x_3, x_4)$,

(C) $\perp(x, y) = \perp(y, x) \quad (x, y \in [0,1])$,

(D) $\perp(\perp(x, y), z) = \perp(x, \perp(y, z)) \quad (x, y, z \in [0,1])$.

A t*–conorm will be called continuous at zero if it satisfies the condition*

(E) for all sequences (x_n) *and* (y_n) *such that* $x_n, y_n \in [0,1] \ (n \in \mathcal{N})$ *and*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0 \ \text{ there holds}$$
$$\lim_{n\to\infty} \perp(x_n, y_n) = 0.$$

For examples see [6] and [7].

In this paper Σ always denotes some σ–algebra of subsets of the set S and $\perp$ a t–conorm continuous at zero.

Definition 2. *A function* $m : \Sigma \to [0,1]$ *with* $m(\emptyset) = 0$ *will be called a* $\perp$*–decomposable measure if*

$$m(A \bigcup B) = m(A) \perp m(B)$$

for all $A, B \in \Sigma$ *such that* $A \bigcap B = \emptyset$.

Theorem 1. *Let $m : \Sigma \to [0,1]$ be a $\perp-$decomposable measure with respect to a continuous at zero $t-$conorm $\perp$. If m is order continuous, then each $m-$Cauchy sequence (E_n), i.e. $m(E_n \Delta E_m) \to 0$ as $n, m \to \infty$, is convergent, i.e. there exists $E \in \Sigma$ such that $m(E_n \Delta E) \to 0$ for $n \to \infty$. Hence the semimetric space (S, Σ, m) is complete.*

Proof. The families

$$V_\sigma = \{E \in \Sigma : m(E) < \sigma\}$$

generate a basis at $\emptyset$ for a FN$-$topology $\Gamma(m)$ on Σ. By Theorem 3.2 from [6] there exists a submeasure η such that $\Gamma(m) = \Gamma(\eta)$, i.e.

$$(*) \qquad m(A_n) \to 0 \iff \eta(A_n) \to 0.$$

Hence for an $m-$Cauchy sequence (E_n) $\eta(E_n \Delta E_m) \to 0$, $n, m \to \infty$. Since by $(*)$ η is also order continuous, it is $\sigma-$subadditive, i.e.

$$\eta(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \eta(A_n)$$

for each sequence $(A_n) \subset \Sigma$. Then by the theorem of Orlicz [5] $\sigma-$algebra Σ is complete with respect to η (also $C-$complete), i.e. there exists set $E \in \Sigma$ such that

$$\eta(E_n \Delta E) \to 0 \ \text{ as } \ n \to \infty.$$

Hence by $(*)$ also

$$m(E_n \Delta E) \to 0 \ \text{ as } \ n \to \infty.$$

Definition 3. *A set $A \in \Sigma$ is an atom of $m, m : \Sigma \to [0,1]$, if $m(A) > 0$ and either $m(B) = 0$ or $m(A \backslash B) = 0$ for every $B \in \Sigma$ such that $B \subset A$. A set $Q \in \Sigma$ is atomless with respect to m if Q contains no atoms of m.*

We shall need in the proof of Theorem 3 the following result from [6].

Theorem 2. *(Saks decomposition). Let $m : \Sigma \to [0,1]$ be an order continuous $\perp-$decomposable measure. Then, for every A in Σ and every $\varepsilon > 0$ there exists a finite number $A_0, A_1, \ldots, A_r$ of pairwise disjoint elements of Σ such that*

(i) $A = \bigcup_{i=1}^{r} A_i$;

(ii) *each* A_i $(i = 0, 1, \ldots, r)$ *is either an atom of* m *and* $m(A_i) > \varepsilon$, *or* $m(A_i) \leq \varepsilon$.

Theorem 3. *Let* $\mathcal{S}$ *be a subfamily of* Σ *such that*

(a) *every* m*-atom is in* $\mathcal{S}$;

(b) *if* $E, F \in \mathcal{S}$ *and* $E \bigcap F = \emptyset$, *then* $E \bigcup F \in \mathcal{S}$;

(c) *if* $E, F \in \mathcal{S}$ *and* $E \subset F$, *then* $F \backslash E \in \mathcal{S}$. *If* $\mathcal{S}$ *has a nonempty interior in* (S, Σ, m), *then* $\mathcal{S} = \Sigma$.

Proof. Since $int\mathcal{S} \neq \emptyset$, there exists $A_0 \in \mathcal{S}$ and $\varepsilon > 0$ such that

$$\mathcal{D} = \{B \in \Sigma : m(A_0 \Delta B) < \varepsilon\} \subset \mathcal{S}.$$

We shall prove that $A \in \Sigma$ and $m(A) < \varepsilon$ implies $A \in \mathcal{S}$. Namely, $A_0 \bigcup A$ and $A_0 \backslash A$ belong to $\mathcal{D}$ and so also $\mathcal{S}$. Hence by (c)

$$A = (A_0 \bigcup A) \backslash (A_0 \backslash A) \in \mathcal{S}.$$

By Theorem 2 there exists a finite number $A_0, A_1, \ldots, A_r$ or pairwise disjoint elements of Σ such that $A = \bigcup_{i=0}^{r} A_i$ and each A_i $(i = 0, 1, \ldots, r)$ is either an atom of m or $m(A_i) < \varepsilon$. If $m(A \bigcap A_i) < \varepsilon$, then $A \bigcap A_i \in \mathcal{S}$.In the opposite case $A \bigcap A_i$ is an m-atom and by (a), $A \bigcap A_i \in \mathcal{S}$. Now, by (b) it follows that $A \in \mathcal{S}$.

Theorem 4. *Let* $\mathcal{S}$ *be a subfamily of* Σ *such that it satisfies (a) – (c) from Theorem 3. If* $\mathcal{S}$ *is a countable union of closed sets in* (S, Σ, m), *then either* $\mathcal{S} = \Sigma$ *or* $\mathcal{S}$ *is of the first category in* (S, Σ, m).

Proof. Since (S, Σ, m) is a complete semimetric space by Theorem 1 we can apply the Bair category theorem and Theorem 3 implies the desired conclusion.

We have now the following generalization of Theorem 3.

Theorem 5. *Let* $(\mathcal{S}_n)$ *be an increasing sequence of subfamilies of* Σ *such that* *(a') every $m-$atom is in $\mathcal{S}_n$ for some n, and that for each n there is a k such that*

(b') if $E, F \in \mathcal{S}_n$ and $E \bigcap F = \emptyset$, then $E \bigcup F \in \mathcal{S}_k$,

(c') if $E, F \in \mathcal{S}_n$ and $E \subset F$, then $F \backslash E \in \mathcal{S}_k$.

If one member of the sequence $(\mathcal{S}_n)$ has a nonempty interior, then there exists a member $\mathcal{S}_s$ such that $\mathcal{S}_s = \Sigma$.

The proof is analogous to the proof of Theorem 2 in [3] and in one part to the proof of Theorem 3 by the use of Theorem 2 instead of the Saks decomposition from [4].

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REZIME

KARAKTERIZACIJA U SMISLU KATEGORIJA NEKIH FAMILIJA SKUPOVA U PROSTORU $\perp$–DEKOMPOZABILNIH MERA

Koristeći neprekidnost t–konorme $\perp$–dekompozabilne mere m u odnosu na pogodno odabranu submeru dokazana je kompletnost semimetričkog prostora (S, Σ, m). Dokazano je da je pod odredjenim pretpostavkama na podfamiliju $\mathcal{S}$ skupova iz (S, Σ, m), ta familija $\mathcal{S}$ ili jednaka Σ ili je prve (Bairove) kategorije.

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