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ORTHOGONAL DECOMPOSITION OF THE INSTANTANEOUS TRANSFORMATION OF THE GAUSSIAN PROCESS

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Abstract

Let $\{\xi(t), 0 \leq t \leq 1\}$ be the mean square continuous Gaussian process. Consider the second order process $\{X(t), 0 \leq t \leq 1\}$ defined by $X(t) = f(\xi(t), t)$ where f is a given non-random function. In the paper the orthogonal decomposition of $\{X(t)\}$ in terms of the orthogonal decomposition of $\{\xi(t)\}$ is determined. The case of Loeve-Karhunen decomposition is also considered.

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1. Some properties of Hermite polynomials

Consider the Hermite polynomial of degree p of the real Gaussian variables $\xi_1, \xi_2, \ldots, E\xi_k = 0$:

(1) $$H_p(\xi_1, \ldots, \xi_p).$$

Some ξ_k in (1) may be equal. Denote by $b_{ij} = E\xi_i\xi_j$. The explicit expression of (1) is

$$\xi_1 \ldots \xi_p - \sum b_{i_1 j_1} \xi_{k_1} \ldots \xi_{k_{P-2}} + \sum b_{i_1 j_1} b_{i_2 j_2} \xi_{k_1} \ldots \xi_{k_p - 4} \ldots,$$

where the first sum is over all the combinations (i_1, j_1) of $\{1, \ldots, p\}$, the second sum is over the disjoint pairs of combinations $(i_1, j_1), (i_2, j_2)$, and so on. For instance

$$H_2(\xi_1, \xi_2) = \xi_1\xi_2 - b_{12},\ H_3(\xi_1, \xi_2, \xi_3) = \xi_1\xi_2\xi_3 - b_{23}\xi_1 - b_{13}\xi_2 - b_{12}\xi_3.$$

Also,

$$H_p(\xi) = H_p\underbrace{(\xi, \ldots, \xi)}_{p\ times} =$$

$$= \xi^p + \sum_{k=1}^{[p/2]}(-1)^k(2k-1)!!\binom{p}{2k} b^k\xi^{p-2k},\ b = E\xi^2. \tag{2}$$

For more details on Hermite polynomials see, for example, [1].

Proposition 1. *If random vectors* $(\xi_1, \ldots, \xi_k), (\eta_1, \ldots, \eta_l), \ldots, (\zeta_1, \ldots, \zeta_m), k + l + \ldots + m = p$ *are independent, then the following factorization is valid*

$$H_p(\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_l, \ldots, \zeta_1, \ldots, \zeta_m) = \tag{3}$$

$$= H_k(\xi_1, \ldots, \xi_k)H_l(\eta_1, \ldots, \eta_l) \ldots H_m(\zeta_1, \ldots, \zeta_m).$$

Proof. It is sufficient to prove that for the two independent vectros $(\xi_1, \ldots, \xi_m)$ and $(\eta_1, \ldots, \eta_n)$ it holds that

$$H_{m+n}(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) = H_m(\xi_1, \ldots, \xi_m)H_n(\eta_1, \ldots, \eta_{n\cdot})$$

The right hand side above is

$$(\xi_1 \ldots \xi_m - \sum_{(i,j)} b_{ij}\xi_{k_1} \ldots \xi_{k_m-2} + \ldots)(\eta_1 \ldots \eta_n - \sum_{(i,j)} c_{ij}\eta_{k_1} \ldots \eta_{k_n-2} + \ldots) =$$

$$= \xi_1 \ldots \xi_m\eta_1 \ldots \eta_n - (\sum_{(i,j)} b_{ij}\xi_{k_1} \ldots \xi_{k_m-2}\eta_1 \ldots \eta_n + \sum_{(i,j)} c_{ij}\eta_{k_1} \ldots \eta_{k_n-2}\xi_1 \ldots \xi_m) + \ldots$$

The expression in the last parenthesis can be put in the form

$$\sum_{(i,j)} a_{ij}\xi_k \ldots \xi_{k'}\eta_l \ldots \eta_{l'},$$

where $a_{ij} = b_{ij}$ if $i, j \in \{k_1, \ldots, k_m\}, a_{ij} = c_{ij}$ if $i, j \in \{l_1, \cdots, l_n\}$ and $a_{ij} = 0$ otherwise, because of the independence of $(\xi_1, \ldots, \xi_m)$ and $(\eta_1, \ldots, \eta_n)$.

It means that this expression is the second term in the Hermite polynomial $H_{m+n}(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$. We shall proceed in the same manner with the following terms of the prod

uct $H_m(\xi_1, \ldots, \xi_m)H_n(\eta_1, \ldots, \eta_n)$. □

Proposition 2.

(4)
$$H_p(\xi_1 + \ldots + \xi_n) =$$
$$= \sum_{\substack{k_1 + \ldots + k_n = p \\ k_i \in \{0, \ldots, p\}}} \frac{p!}{k_1! \ldots k_n!} H_p(\underbrace{\xi_1, \ldots, \xi_1}_{k_1\, times}, \ldots, \underbrace{\xi_n, \ldots, \xi_n}_{k_n\, times}).$$

Proof. Actually on the right hand side in (4) we have the sum of $H_p(\xi_{i_1}, \ldots, \xi_{i_p})$ over all the variations $(i_1, \ldots, i_p)$ with a repetition of elements $\{1, \ldots, p\}$ p at a time. We find, by a more tedious than difficult examination, that each summand on the right hand side is equal to one summand in the explicit expression of $H_p(\xi_1 + \ldots + \xi_n)$, and inversely. After that we obtain (4), since the Hermite polynomial is a symmetric function. □

Remark that using (3) in the case of independent $\xi_1, \ldots, \xi_n$, we have:

(5)
$$H_p(\xi_1 + \ldots + \xi_n) = \sum \frac{p!}{k_1! \ldots k_n!} H_{k_1}(\xi_1) \ldots H_{K_n}(\xi_n).$$

One can prove (5) in another way:

It is evident that

$$H_p(\xi_1 + \xi_2) = \sum_{i_1=0}^{p} \binom{p}{i_1} H_{i_1}(\xi_1) H_{p-i_1}(\xi_2).$$

Then, we obtain

$$H_p(\xi_1 + \ldots + \xi_n) = \sum_{i_1=0}^{p} \binom{p}{i_1} H_{i_1}(\xi_1) H_{p-i_1}(\xi_2 + \ldots + \xi_n) =$$

$$= \sum_{i_1=0}^{p} \binom{p}{i_1} H_{i_1}(\xi_1) \sum_{i_2=0}^{p-i} \binom{p-i_1}{i_2} H_{i_2}(\xi_2) H_{p-i_1-i_2}(\xi_3 + \ldots + \xi_n) = \ldots$$

$$= \dots \sum_{i_1=0}^{p} \sum_{i_2=0}^{p-i_1} \dots \sum_{i_n=0}^{p-(i_1+\dots+i_{n-1})} \frac{p!}{i_1! \dots i_n!} H_{i_1}(\xi_1) \dots H_{i_n}(\xi_n).$$

The last expression is the same as (5).

2. Orthogonal decomposition

Consider the mean square continuous real Gaussian process $\{\xi(t), 0 \leq t \leq 1\}$. Let $\mathcal{H}$ be the linear closure of $\{\xi(t), 0 \leq t \leq 1\}$ and let $\{\eta_1, \eta_2 \dots\}$ be an orthonormal base in the separable Hilbert space $\mathcal{H}$.

Then we have

$$\xi(t) = \sum_{n=1}^{\infty} \varphi_n(t)\eta_n, \; \varphi_n(t) =< \xi(t), \eta_n >= E\xi(t)\eta_n, \tag{6}$$

uniformly in t.

Among the orthogonal decompositions of the form (6), the so-called Loeve-Karhunen decomposition is of special interest. In this case it holds that $(\varphi_i, \varphi_j) = \int_0^1 \varphi_i(t)\varphi_j(t)dt, \; i \neq j$.

Then, we have

$$\eta_n = \int_0^1 \xi(t)\varphi_n(t)dt \tag{7}$$

with the probability one (see, for instance, [2]).

Now, let $f(x,t), -\infty < x < \infty, \; 0 \leq t \leq 1$ be a continuous function. Consider the process $\{X(t), 0 \leq t \leq 1\}$ defined by $X(t) = f(\xi(t), t)$ as the instantaneous transformation of $\{\xi(t)\}$. Suppose that $EX(t) = 0, EX^2(t) < \infty$ for each t.

In this section we shall find the orthogonal decomposition of $\{X(t)\}$ in terms of the orthogonal decomposition of $\{\xi(t)\}$. We shall start from the fact that $\{H_p(\xi(t)), p = 1, 2, \dots\}$ is the complete orthogonal base in the space of all the random variables $Y(t), EY(t) = 0, EY^2(t) < \infty$, measurable with respect to $\xi(t)$. So we have the orthogonal decomposition

$$\begin{aligned} X(t) &= \textstyle\sum_{p=1}^{\infty} a_p(t) H_p(\xi(t)), \\ a_p(t) &= EX(t)H_p(\xi(t)) =< X(t), H_p(\xi(t)) > . \end{aligned} \tag{8}$$

Remark that for the evaluation of $a_p(t)$, it is sufficient to find

$$EX(t)\xi^k(t) = \frac{1}{\sqrt{2\phi b(t)}} \int_{-\infty}^{\infty} f(x,t)x^k \exp\{-\frac{x^2}{2b(t)}\}dx,$$

$$k = 1, 2, \ldots, (b(t) = E\xi^2(t)).$$

From (6) we have

$$H_p(\xi(t)) = H_p(\lim_{n\to\infty} \sum_{k=1}^{n} \varphi_k(t)\eta_k) = \lim_{n\to\infty} H_p(\sum_{k=1}^{n} \varphi_k(t)\eta_k).$$

Keeping in mind that $H_p(a_1\xi_1, \ldots a_p\xi_p) = a_1 \ldots a_p H_p(\xi_1, \ldots, \xi_p)$, it follows from (4)

$$H_p(\sum_{k=1}^{n} \varphi_k(t)\eta_k) =$$

$$\sum_{\substack{k_1 + \ldots + k_n = p \\ k_i \in \{0,\ldots,p\}.}} \frac{p!}{k_1! \ldots k_n!} \varphi_1^{k_1}(t) \ldots \varphi_n^{k_n}(t) H_p(\underbrace{\eta_1, \ldots, \eta_1}_{k_1 \text{times}}, \ldots, \underbrace{\eta_n, \ldots, \eta_n}_{k_n \text{times}}).$$

Since the Gaussian variables $\eta_1, \eta_2, \ldots$ are independent and (3) holds, we obtain

$$(9)\quad H_p(\sum_{1}^{n} \varphi_k(t)\eta_k) = \sum \frac{p!}{k_1! \ldots k_n!} \varphi_1^{k_1}(t) \ldots \varphi_n^{k_n}(t) H_{k_1}(\eta_1) \ldots H_{k_n}(\eta_n).$$

Any two random variables $H_{k_1}(\eta_1) \ldots H_{k_n}(\eta_n)$ and $H_{k_1'}(\eta_1) \ldots H_{k_n'}(\eta_n)$ in (9) differ for at least one $k_i \neq k_i'$. It means that these variables are orthogonal.

It is possible to put (9) in the following form: $H_p(\sum_1^n \varphi_k(t)\eta_k) = \sum_{j=1}^n D_j$, where

$$D_j = \sum_{(j)} \frac{p!}{k_1! \ldots k_j!} \varphi_1^{k_1}(t) \ldots \varphi_j^{k_j}(t) H_{k_1}(t) \ldots H_{k_j}(t).$$

The summation $\sum_{(j)}$ is over all $(k_1, \ldots, k_i)$ such that $k_1 + \ldots + k_j = p$, $k_i \in \{0, \ldots, p\}$ and $k_j \neq 0$. Thus

$$H_p(\xi(t)) = \lim_{n\to\infty} \sum_{j=1}^{n} D_j = \sum_{j=1}^{\infty} D_j.$$

We have finally

Proposition 3. *The process* $\{X(t), 0 \leq t \leq 1\}$ *has the following orthogonal decomposition*

$$X(t) = \sum_{p=1}^{\infty} a_p(t) \sum_{j=1}^{\infty} \Big(\sum_{(j)} \frac{p!}{k_1! \ldots k_j!} \varphi_1^{k_1}(t) \ldots \ldots \varphi_j^{k_j}(t) H_{k_1}(\eta_1 \ldots H_{k_j}(\eta_j)\Big). \tag{10}$$

In the case of the Loeve-Karhunen decomposition one can find an expression analogous to (7). Namely, there is

Proposition 4. *If* $\eta = \int_0^1 \xi(t)\varphi(t)dt$, *then*

$$H_q(\eta) = \int_0^1 \ldots \int_0^1 H_q(\xi(u_1), \ldots, \xi(u_q))\varphi(u_1) \ldots \varphi(u_q) du_1 \ldots du_q.$$

Proof. With some vague but short notations

$$H_q(\zeta(u_1), \ldots, \zeta(u_q)) = \sum b(u,v) \ldots b(u',v')\zeta(w) \ldots \zeta(w'),$$

$$\zeta(u) = \varphi(u)\zeta(u), b(u,v) = E\zeta(u)\zeta(v).$$

We have

$$\int_0^1 \ldots \int_0^1 H_q(\zeta(u_1), \ldots, \zeta(u_q)) du_1 \ldots du_q =$$

$$= \sum E[(\int \zeta(u)du)(\int \zeta(v)dv)] \ldots E[(\int \zeta(u')du')(\int \zeta(v')dv')]$$

$$(\int \zeta(w)dw) \ldots (\int \zeta(w')dw') =$$

$$= \sum E(\eta^2) \ldots E(\eta^2)\eta \ldots \eta = H_q(\eta). \quad \square$$

We obtained for the orthogonal random variables $H_{k_1}(\eta_1) \ldots H_{k_j}(\eta_j)$ in (10) that

$$H_{k_1}(\eta_1) \ldots H_{k_j}(\eta_j) =$$

$$= (\int_0^1 \ldots \int_0^1 H_{k_1}(\xi(u_1), \ldots \xi(u_{k_1}))\varphi_1(u_1) \ldots \varphi_1(u_k) du_1 \ldots du_{k_1}) \ldots$$

$$\ldots (\int_0^1 \ldots \int_0^1 H_{k_j}(\xi(v), \ldots, \xi(v_{k_j}))\varphi_j(v_1) \ldots \varphi_j(v_{k_j}) dv_1 \ldots dv_{k_j}).$$

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REZIME

ORTOGONALNO RAZLAGANJE TRENUTNE TRANSFORMACIJE GAUSOVOG PROCESA

Neka je $\{\xi(t),\ 0 \leq t \leq 1\}$ srednje kvadratno neprekidan Gausov proces. Posmatrajmo proces drugog reda $\{X(t),\ 0 \leq t \leq 1\}$ definisan sa $X(t) = f(\xi(t), t)$ gde je f data neslučajna funkcija. U radu se odredjuje ortogonalno razlaganje za $\{X(t)\}$ u terminima ortogonalnog razlaganja za $\{\xi(t)\}$. Takodje se razmatra slučaj Loev-Karunenovog razlaganja.

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