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* - SEMI - INNER PRODUCT ALGEBRAS OF TYPE(p)

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Abstract

The concept of *-semi-inner product algebras of type(p) is introduced and some properties and results of such algebras are studied. Interesting results about generalized adjoints of bounded linear operators on semi-inner product spaces of type (p) are obtained.

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1. Introduction

Using the concept of semi-inner product space due to Lumer [9], Husain and Malviya [4] introduced the concept of a semi-inner product algebra and extended many results of Ambrose to this class of algebras.

Nath [11] generalized the concept of semi-inner product space to, what he called, generalized semi-inner product space. But he used the same name for another concept in [12]. To avoid this confusion, Abo Hadi [1] called the concept of Nath [11] a semi-inner product space of type(p) and proved, among other results, an analogue of the Riesz Representation Theorem. In the present paper, we shall use this concept to introduce a class of algebras called *-semi-inner product algebras of type(p); they generalize the semi-inner product algebras due to Husain and Malviya [4]. We extend many results of [4] to this new class of algebras. We shall also obtain some interesting results about generalized adjoints of bounded linear operators on semi-inner product spaces of type(p). The concept of a generalized adjoint of a bounded linear operator on a semi-inner product space was considered in [13].

2. Preliminaries

We shall recall some definitions and results from [12].

Definition 2.1. *Let E be a vector space. Let [.,.] be a functional on $E \times E$ defined by*

$$[.,.] : E \times E \longrightarrow K$$

$$< x, y > \longrightarrow [x, y].$$

and satisfying the following conditions:

$$[x + y, z] = [x, z] + [y, z],\ x, y, z \in E. \tag{2.1}$$

$$[\lambda x, y] = \lambda[x, y],\ \lambda \in K \tag{2.2}$$

$$|[x, x]| > 0 \text{ for } x \neq 0. \tag{2.3}$$

$$|[x, y]| \leq [x, x]^{\frac{1}{p}}\ [y, y]^{\frac{p-1}{p}},\ 1 < p < \infty. \tag{2.4}$$

Then we say that [.,.] is a semi-inner product of type(p). A vector space E, together with a semi-inner product of type(p) defined on it , is called a semi-inner product space of type(p).

Remark 2.2. For $p = 2$, it becomes a semi-inner product space due to Lumer [9]. A semi-inner product space E of type(p) is said to be continuous if $[y, x + \lambda y] \to [y, x]$ for all $\lambda \to 0, x, y \in E$.

Theorem 2.3. *A semi-inner product space of type(p) becomes a normed space under* $||x|| = [x, x]^{\frac{1}{p}}$ *and a normed space can be made into a semi-inner product space of type(p).*

3. *-Semi-inner Product Algebras of Type(p)

In this section, we shall introduce the concept of *-semi-inner product algebra of type(p) and study some of the properties of such algebras.

Definition 3.1. *(a) A vector space A is called a semi-inner product algebra of type(p) if*
(SP_1) *A is a Banach algebra, and*
(SP_2) *A is a semi-inner product space of type(p) with the same norm as that in the Banach algebra.*
*(b) A semi-inner product algebra A of type(p) is called a *-semi-inner product algebra of type(p) if corresponding to any $x \in A$, there is an element $x^* \in A$ (called adjoint) satisfying either*

(3.1) $$[xy, z] = [y, x^*z] = [x, zy^*],$$

or

(3.2) $$[z, xy] = [x^*z, y] = [zy^*, x].$$

The following example is adapted from [14]. See also [4].

Example 3.2. Let G be a compact topological group and let $L_p(G), (1 < p < \infty)$, be the space of measurable functions whose pth power is integrable with respect to the Haar measure of G. Then, $L_p(G)$ is a Banach algebra if

$$(f+g)x = f(x) + g(x)$$
$$(fg)x = f(x)g(x)$$
$$(\lambda f)x = \lambda f(x)$$
and $$||f||_p = (\int_G |f|^p dx)^{\frac{1}{p}}.$$

Define $\quad [f, g] = \int_G f(x)|g(x)|^{p-1}(\text{sgn } g(x))dx, \quad f, g \in L_p(G)$

Then, $L_p(g)$ becomes a semi-inner product algebra of type(p). As in [4], we define

$$f^*(x) = \bar{f}(x^{-1}), \; f \in L_p(G),$$

Then, $L_p(G)$ becomes an *-semi-inner product algebra of type(p).

The proof of the following proposition is similar to that of Lemma 3 in [4] and hence, omitted.

Proposition 3.3. *Let A be a $*$-semi-inner product algebra of type(p). If $x \in A$, then $xA = \{0\}$ is equivalent to $Ax = \{0\}$.*

This leads us to the following definition.

Definition 3.4. *Let A be a $*$-semi-inner product algebra of type(p) and $x \in A$. A is called proper if $xA = \{0\} \Rightarrow x = 0$ (equivalently, $Ax = \{0\} \Rightarrow x = 0$).*

Proposition 3.5. *Let A be a proper $*$-semi-inner product algebra of type(p) and $x, y \in A$. Then (a) $x^{**} = x$, and (b) $(xy)^* = y^* x^*$.*

Proof. (a) We know that

$$[z, xy] = [x^* z, y],$$

Replacing x by x^*, we get

$$[x^{**} z, y] = [z, x^* y].$$

Also, we know that

$$[xz, y] = [z, x^* y].$$

So, $[x^{**} z, y] = [xz, y]$ for all y.
Hence $[(x^{**} - x^*) z, y] = 0$ for all y.
Put $y = (x^{**} - x)z$. Then we get

$$||(x^{**} - x)z||^p = 0 \text{ for all } z.$$

From this it follows that $x^{**} = x$, because A is proper.
(b) Similarly, we get $(xy)^* = y^* x^*$.

Now we obtain a characterization of proper $*$-semi-inner algebras of type(p).

Theorem 3.6. *Let A be a $*$-semi-inner product algebra of type(p). Then A is proper if and only if every element of A has a unique adjoint.*

Proof. Suppose A is proper. Let $x \in A$. Let x_1^* and x_2^* be the adjoints of x. Then,

$$[z, xy] = [x*_1 z, y] = [x_2^* z, y] \quad \text{for all } y, z \in A.$$

So, $$[(x_1^* - x_2^*)z, y] = 0 \quad \text{for all } y, z \in A.$$
Put $y = (x_1^* - x_2^*)z$. Then we get

$$||(x_1^* - x_2^*)z||^p = 0 \quad \text{for all } z \in A.$$

Now, it follows that $x_1^* = x_2^*$ because A is proper. The converse follows as in ([4], Theorem 3.1).

Proposition 3.7. *Let A be a proper *-semi-inner product algebra of type(p) and $x \in A, x \neq 0$. Then $xx^* \neq 0, x^*x \neq 0$ and $x^* \neq 0$.*

Proof. Suppose $xx^* = 0$. Then,

$$||yx||^p = [yx, yx] = [yxx^*, y] = 0 \quad \text{for all } y.$$

This implies that $yx = 0$ for all y. Hence $Ax = \{0\}$. Since A is proper, we get $x = 0$. But this contradicts that $x \neq 0$. Similarly, we can prove the other two results.

Notation 3.8. We write E_{oc} to mean the orthogonal complement of a set E, i.e.

$$E_{oc} = \{x \in E : [y, x] = 0,\ y \in E\}.$$

The proof of the following proposition is similar to that of Lemma 3.3 in [4].

Proposition 3.9. *Let A be a (complete) continuous proper *-semi-inner product algebra of type(p) satisfying the inequality*

$$||x + y||^2 + \mu^2||x - y||^2 \leq 2||x||^2 + 2||y||^2, \ 0 < \mu < 1$$

for all $x, y \in A$. Then $xA \subset R \Rightarrow x \in R$.

Proposition 3.10. *Every two-side ideal in a (complete) continuous proper *-semi-inner product algebra A of type(p) which satisfies*

$$||x + y||^2 + \mu^2||x - y||^2 \leq 2||x||^2 + 2||y||^2,\ x, y \in A,\ 0 < \mu < 1,$$

is selfadjoint.

Proof. Let I be a two-side ideal in A. Let $x_1 \in I$ and $x_2 \in I_{oc}$.
Now

$$||x_1x_2||^p = [x_1x_2, x_1x_2] = [x_1^*x_1x_2, x_2] = 0.$$

The rest of the proof is as in Lemma 3.4 of [4].

The following results follow as in [4].

Proposition 3.11. *If R is a right ideal in a proper *-semi-inner product algebra of type(p), then the right ideal generated by R^n is R, where R^n stands for the set of elements of the form $x_1x_2\ldots x_n$, $x_1, x_2 \ldots, x_n \in R$.*

Proposition 3.12. *Let A be a continuous proper *-semi-inner product algebra of type(p) satisfying the inequality*

$$||x+y||^2 + \mu^2||x-y||^2 \leq 2||x||^2 + 2||y||^2,$$

$0 < \mu < 1$, $x, y \in A$. *Then the set D of all the elements of the form $x_1y_1 + \ldots + x_ny_n$ is dense in A.*

Proposition 3.13. *Let A be as in Proposition 3.12 and I a right deal in A. Then, $L(I) = \{x; xI = (0)\}$ is the orthogonal complement of I^* in A.*

Theorem 3.14. *Let A be as in Proposition 3.12. Also, let A be a strictly convex space in which the weak convergence in the second component is finer than the norm topology. Further assume that $[x, y] = [y^*, x^*]$ holds for $x \in A$ and $y \in D$ (D as defined in Proposition 3.12). Then $||x|| = ||x^*||$ and the map $x \longrightarrow x^*$ is continuous.*

4. Existence of Idempotents

In this section we shall study the existence of idempotents in *-semi-innner product algebras of type(p) and prove that a *-semi-inner product algebra of type(p), under certain restrictions, contains a maximal family of doubly orthogonal primitive self-adjoint idempotents.

Definition 4.1. *An element e in a *-semi-inner product algebra of type(p) is called an idempotent if $0 \neq e = e^2$. The element e is called self-adjoint if $e = e^*$.*

Henceforth, we assume that a *-semi-inner product algebra of type(p) satisfies $[x, y] = [y^*, x^*]$ and

$$(\alpha x + \beta y)^* = \bar{\alpha} x^* + \bar{\beta} y^*, \ \alpha, \beta \in K.$$

Proposition 4.2. *Let A be a proper *-semi-inner product algebra of type(p). Let x be a self-adjoint element of A whose norm as a left multiplication operator is 1. Then the sequence* x^{2n} *converges to a non - zero self-adjoint idempotent.*

Proof. Following Loomis [8], page 101, we proceed as follows:

Let $|||y|||$ be the operator norm of y defined by

$$|||y||| = \sup_z (||yz|| \ / \ ||z||).$$

Since

$$||yz|| \leq ||y|| \ ||z||$$

we have $|||y||| \leq ||y||$. Since x is a self-adjoint element such that $|||x||| = 1$, $|||x^n||| = 1$ and hence $||x^n|| \leq 1$ for all n. If $m > n$ and if they are both even, then

$$\begin{aligned} &[x^m, x^n] \leq |[x^m, x^n]| \leq [x^m, x^m]^{\frac{1}{p}} [x^n, x^n]^{\frac{p-1}{p}}, \quad 1 < p < \infty \\ &\leq ||x^m|| \ ||x^n||^{p-1} = ||x^{m-n} x^n|| \ ||x^n||^{p-1} \\ &= \frac{||x^{m-n} x^n||}{||x^n||} ||x^n||^p \leq |||x^{m-n}||| \ ||x^n||^p. \end{aligned}$$

Hence

$$[x^m, x^n] \leq |||x^{m-n}||| \ ||x^n||^p = [x^n, x^n].$$

So

$$[x^m, x^n] \leq [x^n, x^n].$$

Next

$$[x^m, x^m] = ||x^m||^p = ||x^{\frac{m-n}{2}} x^{\frac{m+n}{2}}||^p.$$

Put $2r = m - n$. Then

$$[x^m, x^m] = ||x^r x^{n+r}||^p = \frac{||x^r x^{n+r}||^p}{||x^{n+r}||^p} ||x^{n+r}||^p \leq$$

$$\leq |||x^r|||^p \ ||x^{n+r}||^p = [x^{n+r}, \ x^{n+r}] = [x^{*r} x^{m-r}, x^n] = [x^m, x^n].$$

Hence,

$$[x^m, x^m] \leq [x^m, x^n].$$

Thus

$$1 \leq [x^m, x^m] \leq [x^m, x^n] \leq [x^n, x^n] \leq \ldots \leq [x^2, x^2]$$

and $[x^m, x^n]$ has a limit $L \geq 1$ as $m, n \to \infty$ through even integres. Hence, we have

$$\lim ||x^m - x^n||^p = \lim[x^m, x^n, x^m - x^n] =$$

$$= \lim[x^m, x^m - x^n] - \lim[x^n, x^m - x^n] =$$

$$= \lim[x^{*m} - x^{*n}, x^{*m}] - \lim[x^{*m} - x^{*n}, x^{*n}] =$$

$$= \lim[x^m, x^m] - \lim[x^n, x^m] - \lim[x^m, x^n] + \lim[x^n, x^n]$$

which tends to zero as $m, n \to \infty$. And x^n converges to a self adjoint element e with $||e|| \geq 1$, since x^{2n} converges both to e and to e^2, it follows that e is idempotent.

Corollary 4.3. *Any left (or right) ideal I in a proper *-semi-inner product algebra of type(p) contains a non-zero self-adjoint idempotent.*

Definition 4.4. *(a) The idempotents e and f of an *-semi-inner product algebra A of type(p) are called doubly orthogonal if* $[e, f] = 0$ *and* $ef = fe = 0$.
(b) An idempotent is said to be primitive if it can not be expressed as the sum of doubly orthogonal idempotents.

The following results follow as in [2] and [4].

Proposition 4.5. *Let A be a proper *-semi-inner product algebra of type(p). Let e be an idempotent in A and* $R = eA$ *the right ideal in A. If* $R = R_1 + \ldots + R_n$, *each* R_i *being a right ideal and if* $e = e_1 + \ldots + e_n, e_i \in R_i$, *then the* e_i *is a self -adjoint idempotent.*

Proof. Similar to that in [2].

Proposition 4.6. *Let A, e and R be as in Proposition 4.5. If e can be expressed as a finite sum of doubly orthogonal self adjoint idempotent, say,* $e = e_1 + \ldots + e_n$, *and if we define* R_i *by* $R_i = e_i A$, *then R is the direct sum of right ideals* R_i.

Proof. Similar to that in [2].

Theorem 4.7. *Let A, e and R be as in Proposition 4.5. Then R is minimal if and only if e is primitive.*

Proof. Similar to that in [4], page 103.

Theorem 4.8. *Let A and e be as in Proposition 4.5. Then e is the sum of a finite number of doubly orthogonal primitive self-adjoint idempotents.*

Proof. Following Ambrose [2], we can write $e = e_1 + \ldots + e_n$ where $e_i, e_2, \ldots, e_n$ are self-adjoint idempotents. Now,

$$||e||^p = [e_1 + e_2 \ldots + e_n, e_1 + e_2 + \ldots + e_n] =$$

$$= [e_1, e_1] + [e_2, e_2] + \ldots + [e_n, e_n] =$$

$$= ||e_1||^p + ||e_2||^p + \ldots + ||e_n||^p \geq n,$$

since

$$||e_i||^p = [e_i, e_i] = [e_i^2, e_i] \leq ||e_i^2|| \, ||e_i||^{p-1} \leq ||e_i||^{p+1}$$

or

$$||e_1|| \geq 1, i = 1, 2, \ldots, n.$$

This shows that the process of splitting e must terminate at some finite stage.

Theorem 4.9. *Let A be a proper *-semi-inner product algebra of type(p). Then A contains a maximal family of doubly orthogonal primitive self-adjoint idempotents.*

Proof. By Corollary 4.3, A contains self-adjoint idempotents. By Theorem 4.8, A contains a finite family of doubly orthogonal primitive self-adjoint idempotents. Hence, by Zorn's Lemma, A contains a maximal family of doubly orthogonal primitive self-adjoint idempotents.

5. Bounded Linear Operators and Generalized Adjoint Operators

In this section we shall consider a concept called the generalized adjoint of a bounded linear operator on a semi-inner product space of type(p) and obtain some interesting results.

The concept of the generalised adjoint of a bounded linear operator was considered in [13].

Notation 5.1. Let E be a (complete) continuous semi-inner product space of type(p), satisfying the inequality

$$||u+v||^2 + \mu^2||u-v||^2 \leq 2||u||^2 + 2||v||^2, \; 0 < \mu < 1.$$

Let T be a bounded linear operator on E. Define $g_y(x) = [Tx, y]$. Then g_y is a continuous linear functional. Hence (by the analogue of the Riesz representation theorem), there exists a unique vector T^*y such that

$$[Tx, y] = g_y(x) = [x, T^*y], \quad x \in E$$

We call T^* the generalised adjoint of T.

Remark 5.2. T^* is not linear.

Theorem 5.3.

$$(i) \quad ||T|| = ||T^*||^{p-1}$$
$$(ii) \quad ||T^*T||^{p-1} = ||T||^p$$

Proof.

$$\begin{aligned} (i)\ ||Ty||^p &= [Ty, Ty] \\ &= [y, T^*Ty] \\ &\leq ||y||\ ||T^*Ty||^{p-1} \\ &\leq ||y||\ ||T^*||^{p-1}||Ty||^{p-1} \end{aligned}$$

Hence,

(5.1) $$||Ty|| \leq ||y||\ ||T^*||^{p-1} \text{ for all } y$$

$$\Rightarrow ||T|| \leq ||T^*||^{p-1}.$$

Next,

$$\begin{aligned} ||T^*y||^p &= [T^*y, T^*y] \\ &= [TT^*y, y] \\ &\leq ||TT^*y||\,||y||^{p-1} \\ &\leq ||T||\,||T^*y||\,||y||^{p-1} \end{aligned}$$

$$||T^*y||^{p-1} \leq ||T||\,||y||^{p-1},$$

Hence,

(5.2) $$||T^*||^{p-1} \leq ||T||$$

From (5.1) and (5.2) we get

$$||T|| = ||T^*||^{p-1}$$

$$\begin{aligned} (ii)\ ||T^*T|| &= \sup\{||T^*T(x)||,\ ||x|| \leq 1\} \\ &\leq ||T^*||\sup\{||Tx||, ||x|| \leq 1\} \\ &\leq ||T^*||\,||T||. \end{aligned}$$

Hence,

$$||T^*T||^{p-1} \leq ||T^*||^{p-1}\,||T||^{p-1}$$

Using (i) we get

(5.3) $$||T^*T||^{p-1} \leq ||T||^p$$

Now,

$$\begin{aligned} ||T||^p &= \sup\{||Tx||^p;\ ||x|| \leq 1\} \\ &= \sup\{[Tx, Tx];\ ||x|| \leq 1\} \\ &= \sup\{[x, T^*Tx];\ ||x|| \leq 1\} \\ &\leq \sup\{||x||\,||T^*Tx||^{p-1}; ||x|| \leq 1\} \\ &\leq \sup\{||T^*Tx||^{p-1}; ||x|| \leq 1\}, \end{aligned}$$

Hence,

(5.4) $$||T||^p \leq ||T^*T||^{p-1}$$

From (5.3) , (5.4) we get

$$||T||^p = ||T^*T||^{p-1}.$$

Let A be a (complete) continuous and proper *-semi-inner product algebra of type(p) satisfying the inequality

$$||u+v||^2 + \mu^2||u-v||^2 \leq 2||u||^2 + 2||v||^2, \ 0 < \mu < 1.$$

If $\beta(A)$ stands for the space of bounded linear operators on A, we define

$$\beta_L(A) = \{T_x \in \beta(A) : T_x y = xy\}$$

Lemma 5.4. $T_x* = T_x^*$

Proof.

$$[xy, z] = [T_x y, z] = [y, T_x^* z]$$

$$[y, x^* z] = [y, T_x^* z]$$

$$[y, T_x^* z] = [y, T_x^* z]$$

$\Rightarrow T_x^* = T^* x$ by the Riesz representation theorem (uniqueness).

Theorem 5.5.

$$(1) \qquad T_x^{**} = T_x$$
$$(2) \quad (T_x T_y)^* = T_y^* T_x^*$$

Proof. Follows easily using Lemma 5.4.

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REZIME

* - POLU - SKALARNI PROIZVOD NA ALGEBRAMA TIPA(p)

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