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AN ALGORITHM FOR CAYLEY TABLES OF ALGEBRAS

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Abstract

The paper presents an algorithm for the determination of all nonisomorphic algebras of n elements in some varieties.

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Consider the variety of semigroups. Suppose that the elements of semigroups are $1, 2, ..., n$. We can determine all nonisomorphic semigroups of n elements in the following way.

Let the matrix M represent a semigroup, i.e. $M(i,j) = i * j$, $1 \leq i,j \leq n$. Because we consider only nonisomorphic semigroups, we can devide all semigroups into two classes:

- bands, i.e. the semigroups for which $x * x = x$ for each x, $1 \leq x \leq n$,
- nonbands, for which, because of isomorpism, we can suppose that $1 * 1 = 2$.

The elements of a matrix M are here devided into three types: freely defined elements, consequences of defined elements and undefined elements. Besides, each element $M(i,j)$ of a matrix has an ordinal number $Z(i,j)$ which corresponds to the order of its defining. We define only the value of freely defined elements. The value of consequences of defined elements are determined by applying the condition $x*(y*z) = (x*y)*z$ for some x, y and z. We choose the ordinal number $Z(i,j)$ of freely defined elements $M(i,j)$ in the following way: $Z(i,j) = n(i+j)+j$. The ordinal number of each undefined element is 0. Freely defined elements and their consequences have the same ordinal number.

We determine the consequences of the element $i*j$ (i.e. $M(i,j)$) by applying the semigroup condition $(i*j)*k = i*(j*k)$ and $k*(i*j) = (k*i)*j$ for $1 \le k \le n$. The undefined element $x*y$ becomes the consequence of defined elements in the following cases:

1. $x = i*j$; $y = k$, $j*k$ and $i*(j*k)$ are defined elements
2. $x = i$, $y = j*k$; $(i*j)*k$ is a defined element,
3. $x = k$, $y = i*j$; $k*i$ and $(k*i)*j$ are defined elements,
4. $x = k*i$; $y = j$; $k*(i*j)$ is a defined element.

$Z(x,y)$ is equal to the maximum of the ordinal numbers of the elements which determine $x*y$, because $M(x,y)$ is a consequence of the last freely defined element among elements which define it. For example, in case 1., $Z(x,y) = max\{Z(i,j), Z(j,k), Z(i,j*k)\}$ is satisfied.

On the other hand, if $j*k$, $i*(j*k)$ and $(i*j)*k$ are defined elements and $(i*j)*k \neq i*(j*k)$ (analogously if $k*(i*j) \neq (k*i)*j$), then we get a contradiction. Therefore, the given matrix cannot be completed to represent a semigroup.

The function Z determines the order of defining and examining of elements. In our case, we examine elements in the following order: $(1,1),(2,1)$,-$(1,2),(3,1)$, $(2,2),(1,3),(4,1),\ldots$ Our algorithm examines by the "back-tracking" method all possible matrices (with the above three types of elemnats) and elimenates those marices which cannot be completed to represent semigroups. We describe the algorithm by a sequence of steps.

STEP 1. (Initial step). Let $M(i,j) = i$ and $Z(i,i) = n(1+1)+1 = 2n+1$, $(1 \leq i \leq n)$ for bands and $M(1,1) = 2$ and $Z(1,1) = 2n+1$ for nonbands. In both cases let $Z(i,j) = 0$ otherwise.

STEP 2. Examine the next element $M(i,j)$ of the matrix M. If $M(i,j)$ is a consequence of defined elements (i.e. $Z(i,j) \neq 0$ and $Z(i,j) \neq n(i+j)+j$), then go to Step 3. If $M(i,j)$ is an undefined element (i.e. $Z(i,j) = 0$), then $M(i,j)$ becomes a freely defined element with the value $M(i,j) = 1$ and its ordinal number is $Z(i,j) = n(i+j)+j$. Go to the next step.

STEP 3. Determine the consequences of the elements $M(i,j)$ (in the way described above). If there is a contradiction, go to STEP 4.; otherwise go to STEP 5.

STEP 4. Let $M(i,j)$ be the last freely defined element. If this element is $M(1,1)$, then finish the algorithm. In the oposite case, $M(i,j)$ leads to a contradiction. All elements with the same odrinal number as $M(i,j)$ (i.e. the consequences of the element $M(i,j)$ become undefined elements (except $M(i,j)$). If $M(i,j) < n$, then add 1 to $M(i,j)$ and go to STEP 3. If $M(i,j) = n$, then let $M(i,j)$ be an undefined element and go to STEP 4.

STEP 5. If there are undefined elements in the matrix M, then go to STEP 2. If all elements in the matrix M are defined, check wether M presents a semigroup (the condition $x*(y*z) = (x*y)*z$ is not examined for all x, y, z during the generation of the semigroup). If M presents a semigroup which is representetive of its class of isomorphic semigroups then write the obtained matrix. In each case go to STEP 4. (As if there were a contradiction).

We can examine wether the obtained semigroup is representative of its class of semigroups in the follwing way. Let $<$ be an order relation between all semigroups isomorphic to the semigroup presented by M (for example, lexicographic order) and $M_1 < M_2 < \ldots < M_t$ are all these semigrous. If $*$ is the semigroup operation represented by M, and $*_i$ is the semigroup operation of M_i, then $x *_i y = (\sigma^{-1}(x) * \sigma^{-1}(y))$, where σ is a permutation of elements $1, 2, \ldots, n$. We consider only the matrices M_i for wich $M(1,1) = M_i(1,1)$ is satisfied. If $M = M_i$, then M is a representative of its class of semigrouups. If we consider the representatives of the classes of nonisomorphic and nonantiisomorphic semigroups then we must include in order relation antiisomorphic semigroups also, defined in the following way $x *_i y = (\sigma^{-1}(y) * \sigma^{-1}(x))$. We can stop this examination if we find a semigroup represented by matrix M_i for which $M_i < M$, because in this case M is not a representative of its class of semigroup.

A program on PL-language is written on the basis of this algorithm and it is implemented on the IBM 4341 computer. The time required to the semigroups of orders 2 through 5 totaled less than 10 minutes, while those of order 6 required 50 hours. The obtained data are presented in the following table.

number of elements of semigroups	2	3	4	5	6
number of non isomorphic semigroups	5	24	188	1915	28634
number of nonisomorphic and nonantiisomorphic semigroups	4	18	126	1160	15873

We list the representatives of the clases of non isomorphic and non antiisomorphic semigroups containing n elements for $2 \leq n \leq 4$. We also give Caley tables of all bands of 5 elements. The complete list of semigroups of n elements, for $2 \leq n \leq 5$, can be found in [1]. For some other listings see [2] - [6].

If we have more than one law in a variety of the type 2. for example if we determine comutative semigroups, then in Step 3 (and in the part of Step 5) we have to consider universal laws $x*(y*z) = (x*y)*z$ and $x*y = y*x$ (instead of associativity condition only).

The algorithm presentad here can be slightly modified and applied to the varietes with binary fundamental operations. We assume that we do not know Cayley tables for only one operation in a given variety. This algorithm is especially suitable if the unknown operation is cancellative. Of course, in the case of varieties with more than one binary operation, the list of consequences of $i*j$ has to be extended. In general terms, if

$$(1) \qquad F_\alpha(i*j, k_1 ..., k_n) = F_\beta(i, j, k_1, ..., k_n)$$

is an identity and the values of F_α or F_β can be verified we have:

1. F_α can be calculated and F_β reduced to a form $a*b$ (all possible reducements);

2. F_β can be calculated and F_α reduced to a form $a * b$ (all possible reducements), then $a*b$ is aconsequence of $i*j$ if it was not previously calculated. If $a * b$ is known, then there are two cases:

   - The identity (1) holds. Then we continue the search of the consequences of $i * j$;
   - The identity (1) does not hold, then $i*j$ leads to a contradiction and the search has the following form:
     If $i * j < n$, then increase $i * j$ by 1 and continue the saerch;
     If $i * j = n$, then go back to the previously freely defined element which should be increased by 1 etc.

In general, if we have a variety with n-ary opreations the algorithm above should be modified. For example, if we have a ternary operation, then we are dealing with three dimensional matrices and we have to define the order of elements in this matrix.

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REZIME

JEDAN ALGORITAM ZA KEJLIJEVE TABLICE ALGEBRI

Dat je algoritam za nalaženje svih neizomorfnih konačnih algebri datog tipa. Rad algoritma demonstriran je u slučaju polugrupa reda manjeg od 6.

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SEMIGROUPS OF ORDER 2

11	11	21	22
12	22	12	22

SEMIGROUPS OF ORDER 3

111	111	111	111	111	111	211	211	212
121	122	122	121	123	222	122	122	121
113	123	133	333	333	333	122	123	212

213	221	221	222	222	222	222	223	231
123	222	222	222	222	222	222	223	312
333	123	223	221	222	223	333	333	123

SEMIGROUPS OF ORDER 4

1111	1111	1111	1111	1111	1111	1111	1111	1111	1111
1211	1211	1211	1212	1222	1222	1222	1222	1222	1222
1131	1133	1133	1133	1232	1233	1233	1232	1234	1333
1114	1134	1144	1234	1224	1234	1244	1444	1444	1444

1111	1111	1111	1111	1111	1111	1111	1111	1111	1111
1211	1211	1224	1234	1224	1212	1212	1214	1212	1211
1131	1134	1234	1234	1334	3333	3333	3333	3333	3333
4444	4444	4444	4444	4444	1214	1234	1214	1414	3334

1111	1111	1111	1111	1133	1111	2111	2111	2111	2111
1212	1211	1214	1234	2244	2222	1222	1222	1222	1222
3333	3333	3333	3333	1133	3333	1222	1222	1222	1222
3434	4444	4444	4444	2244	4444	1222	1223	1224	1234

2111	2111	2111	2111	2111	2111	2112	2112	2112	2112
1222	1222	1222	1222	1222	1222	1221	1221	1221	1221
1223	1232	1233	1233	1233	1234	1221	1221	1231	1231
1234	1224	1233	1234	1244	1243	2112	2113	2112	2142

2112	2112	2114	2114	2122	2124	2133	2133	2133	2133
1221	1221	1224	1224	1211	1214	1233	1233	1233	1233
1234	1234	1224	1234	2122	2124	3333	3333	3333	3333
2142	2143	4444	4444	2122	4444	3333	3334	4433	4444

2134	2134	2134	2112	2134	2143	2143	2211	2211	2211
1234	1234	1234	1221	1234	1234	1234	2222	2222	2222
3333	3333	3333	4334	4334	4312	4321	1233	1233	1233
4433	4434	4444	3443	3434	3421	3412	1233	1234	1244

2211	2212	2212	2212	2212	2212	2212	2212	2212	2212
2222	2222	2222	2222	2222	2222	2222	2222	2222	2222
1234	1231	1232	1232	1231	1232	1234	1234	1234	1232
2244	2212	2222	2224	2242	2242	2241	2242	2244	4444

2212	2214	2212	2212	2211	2211	2211	2211	2212	2212
2222	2224	2222	2222	2222	2222	2222	2222	2222	2222
1234	1234	2232	2232	2233	2233	2234	2233	2232	2232
4444	4444	1214	1224	2233	2234	2234	2244	2212	2222

2212	2212	2212	2212	2214	2214	2212	2212	2214	2222
2222	2222	2222	2222	2224	2224	2222	2222	2224	2222
2232	2234	2232	2234	2234	2234	2232	2234	2234	2211
2224	2222	2242	2244	2224	2244	4444	4444	4444	2211

2222	2222	2222	2222	2222	2222	2222	2222	2222	2222
2222	2222	2222	2222	2222	2222	2222	2222	2222	2222
2211	2211	2211	2212	2212	2212	2221	2221	2221	2222
2212	2221	2222	2221	2222	2224	2212	2213	2222	2222

2222	2222	2222	2222	2222	2222	2222	2222	2222	2224
2222	2222	2222	2222	2222	2222	2222	2222	2222	2224
2222	2232	2232	2233	2233	2212	2222	2232	2234	2214
2224	2224	2233	2234	2244	4444	4444	4444	4444	4444

2224	2224	2222	2222	2222	2223	2224	2233	2233	2234
2224	2224	2222	2222	2222	2222	2224	2233	2233	2234
2224	2234	3333	3333	3333	3333	3334	3333	3333	3333
4444	4444	3333	3334	4444	4444	4444	3334	4444	4434

2234	2234	2212	2311	2312	2314
2234	2234	2222	3122	3123	3124
3333	3342	4434	1233	1231	1234
4444	4423	4444	1234	2312	4444

BANDS OF ORDER 5

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12111	12111	12111	12111	12111	12111	12111	12111	12111	12111
11311	11311	11311	11313	11333	11333	11333	11333	11333	11333
11141	11144	11144	11144	11343	11344	11344	11343	11345	11444
11114	11145	11155	11345	11335	11345	11355	11555	11555	11555

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12112	12112	12112	12112	12112	12112	12112	12112	12112	12122
11313	11331	11333	11333	11333	11333	11331	11341	11331	11333
11144	11341	11343	11344	11444	11444	11341	11341	11441	12344
12345	12115	12335	12345	12345	12345	15115	15115	15115	12345

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12122	12111	12111	12111	12111	12155	12111	12222	12222	12222
11333	11311	11311	11335	11345	11341	11335	12322	12322	12322
12344	11141	11145	11345	11345	11341	11445	12242	12244	12244
12355	55555	55555	55555	55555	55555	55555	12225	12245	12255

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12222	12222	12222	12222	12222	12222	12222	12222	12222	12222
12323	12333	12333	12333	12333	12333	12333	12322	12322	12335
12244	12343	12344	12344	12343	12345	12444	12242	12245	12345
12345	12335	12345	12355	12555	12555	12555	15555	15555	15555

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12222	12222	12222	12222	12222	12222	12222	12222	12222	12222
12345	12335	12323	12323	12325	12323	12322	12323	12322	12325
12345	12445	14444	14444	14444	14444	14444	14444	14444	14444
15555	15555	12325	12345	12325	12525	14445	14545	15555	15555

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12222	12225	12225	12225	12225	12245	12245	12225	12225	12345
12345	12325	12335	12345	12335	12345	12345	12325	12345	12345
14444	12245	12345	12345	12445	12245	12445	14445	14445	12345
15555	55555	55555	55555	55555	55555	55555	55555	55555	55555

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12244	12222	12225	12111	12111	12111	12111	12112	12112	12112
13355	13333	13335	11313	11313	11315	11313	11313	11313	11341
12244	14444	14445	44444	44444	44444	44444	44444	44444	44444
13355	15555	55555	11315	11345	11315	11515	12315	12345	12115

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12112	12112	12111	12111	12111	12111	12112	12111	12111	12111
11343	11341	11311	11313	11344	11345	11344	11311	11315	11344
44444	44444	44444	44444	44444	44444	44444	44444	44444	44444
12345	15115	44445	44545	44445	44545	45445	55555	55555	55555

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12111	12115	12245	12345	12245	12122	12122	12122	12122	12122
11345	11341	12345	12345	13345	33333	33333	33333	33333	33333
44444	44444	44444	44444	44444	12142	12144	12145	12144	12142
55555	55555	55555	55555	55555	12125	12145	12145	12155	12325

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12122	12125	12125	12122	12122	12122	12125	12125	12122	12122
33333	33333	33333	33333	33333	33333	33333	33333	33333	33333
12144	12145	12145	12344	12345	12344	12345	12345	12142	12145
12345	12125	12155	12345	12345	12355	12125	12155	15155	15155

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12122	12122	12145	12122	12122	12122	12122	12125	12125	12145
33333	33333	33333	33333	33333	33333	33333	33333	33333	33333
12342	12345	12145	14144	12142	12345	14144	12145	12345	12145
15155	15155	12145	15155	35355	35355	35355	55555	55555	55555

11111	11111	11144	11144	11144	11111	11111	11111	11111	11111
12145	12125	12144	12144	12345	12112	12111	12111	12111	12111
33333	33333	33355	33355	33355	33333	33333	33333	33333	33333
12545	14145	11144	14144	14144	33344	33344	33345	33344	33343
55555	55555	33355	35355	35355	12345	33345	33345	33355	55555

11111	11111	11111	11111	11111	11111	11111	11111	11111	11111
12111	12125	12112	12112	12112	12112	12115	12112	12112	12111
33333	33333	33333	33333	33333	33333	33333	33333	33333	33333
33345	34345	44444	44444	44444	44444	44444	44444	44444	44444
55555	55555	12115	12145	12335	12345	12115	15115	35335	55555

11111	11111	11111	11111	11111
12115	12144	12145	12345	22222
33333	33333	33333	33333	33333
44444	44444	44444	44444	44444
55555	55555	55555	55555	55555