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FREE POLY-ALGEBRAS

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Abstract

In this paper we consider two kinds of free structures: "free" and "strongly free" poly-algebras.

We shall give a complete description of free objects over the class of regular poly-algebras and show the existence of strongly free object in the class of α- bounded regular objects, where α is a non-empty cardinal. If α is finite, two strongly free objects with the same basis are isomorphic, and in this case we have a complete description of strongly free objects. But, if α is infinite, then there exist non-isomorphic strongly free objects with the same basis.

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1. Introduction

Let A be a non-empty set. A *poly-operation of arity n* (or a *poly-n-operation*) is a mapping from A^n into the family $\mathcal{P}(A)$ of all subsets of A. Roughly speaking, a *poly-algebra* is a non-empty set with some poly-operations. In the literature they are also known as *multi-algebras* or *hyper-algebras*. If the results of operations are always non-empty sets, the poly-algebra is called *regular*. If we do not make any distinction between the singleton $\{x\}$ and element x, then every universal algebra and every partial algebra is a special poly-algebra.

Poly-algebras with only one poly-n-operation we call *poly-n-groupoids*. In the literature, poly-2-groupoids are called *multi-groupoids* (see [1]). Poly-groups, hyper-groups and n-ary structures are some special multi-groupoids (see [2],[5]).

In this paper we start from two kinds of homomorphisms and consequently, consider two kinds of free poly-algebras. To make distinction between them, we will use the names "free" and "strongly free" poly-algebra. Both notions coincide with the usual notion of free algebra in the case of the class of universal algebras. (Note, that in [4] one more kind of homomorphism and the corresponding "freeness" is considered .) Here, we shall give a complete description of free objects over the class of regular poly-algebras of fixed type. From that result it can be deduced that every non-empty set is the basis of non-isomorphic free objects. Concerning strongly free objects,the situation is the following: the class of all regular poly-algebras does not admit any strongly free objects (see [4]). Here, we show the existence of strongly free object in the class of so called *α-bounded regular objects*, where α is a non-empty cardinal. If α is finite, two strongly free objects with the same basis are isomorphic, and in this case we have a complete description of strongly free objects. But, if α is infinite, then the situation is unusual as in the case of free objects. Namely, there exist non-isomorphic strongly free objects with the same basis.

2. Basic notions and notations

2.1. Poly-algebras

Let $\mathcal{F}$ be a non-empty disjoint union of some sets $\mathcal{F}_n$, $n \in N$. The set $\mathcal{F}$ we call a *type*, and the elements of $\mathcal{F}$ we call *functional symbols*. If $A \neq \emptyset$, then any mapping from A^n into the family $\mathcal{P}(A)$ of all subsets of A is called a *poly-n-operation on A*. Let $\mathcal{A}$ be a mapping from $\mathcal{F}$ into the family of poly-operations on some set A , such that for all $n \in N$, $f \in \mathcal{F}_n$, $\mathcal{A}(f)$ is a poly-n-operation. Then, for $\mathcal{A}(f)$ we say that it is an *interpretation of the functional symbol f in $\mathcal{A}$*, and the mapping $\mathcal{A}$ we call a *poly-algebra of type $\mathcal{F}$ (or poly-$\mathcal{F}$-algebra) with the carrier A*. Interpretation of the symbol f in $\mathcal{A}$ we shall denote by $f^{\mathcal{A}}$. Instead of "poly-algebra", we shall often say an "object".

Let $\mathcal{F}$ be some fixed type of poly-algebras, and A some non-empty set. In the sequel, we use the notation $\mathcal{F}(A)$ for the set of all poly-$\mathcal{F}$-algebras with the carrier A. We shall say that $\mathcal{A} \in \mathcal{F}(\mathcal{A})$ is *regular* iff $f^{\mathcal{A}}(x) \neq \emptyset$, for all $n \in N$, $f \in \mathcal{F}_n$, $x \in A^n$, . The class of all regular poly-$\mathcal{F}$-algebras will be denoted by $\mathcal{R}eg(\mathcal{F})$. We have already noted, that any universal algebra can be considered as a special poly-algebra $\mathcal{A}$. The class of all such poly-$\mathcal{F}$-algebras we denote by $\mathcal{U}al(\mathcal{F})$.

Let α be a cardinal number, $\alpha \neq 0$. The class $\mathcal{R}eg[\mathcal{F}, \alpha]$ of *α-bounded regular poly-$\mathcal{F}$-algebras* is defined by

$$\mathcal{R}eg[\mathcal{F}, \alpha] = \{\mathcal{A} \in \mathcal{R}eg(\mathcal{F}) : (\forall n \in N)(\forall f \in \mathcal{F}_n)(\forall a \in A^n) \mid f^{\mathcal{A}}(a) \mid \leq \alpha\}.$$

Of course, $\mathcal{R}eg[\mathcal{F}, 1] = \mathcal{U}al(\mathcal{F})$.

2.2. Terms

The notions of *$\mathcal{F}$-terms over X* $(X \neq \emptyset, \mathcal{F} \bigcap X = \emptyset)$ is defined in the usual way. The set of all $\mathcal{F}$-terms over X will be denoted by $X(\mathcal{F})$ and it is in fact the carrier of the absolutely free $\mathcal{F}$-algebra $\mathcal{X}(\mathcal{F})$ with the basis X. Thus,

$$f^{\mathcal{X}(\mathcal{F})}(t_1, t_2, ..., t_n) = f(t_1, t_2, ..., t_n),$$

for every $f \in \mathcal{F}_n$, $t_i \in X(\mathcal{F})$. This algebra is called the *algebra of $\mathcal{F}$ -terms over X*, and it will be usually denoted by $X(\mathcal{F})$, as its carrier.

The *rang (or complexity)* of some term $t \in X(\mathcal{F})$ is defined in the following way:

- If $t \in X \bigcup \mathcal{F}_0$ then rang $(t) = 0$,
- $rang(f(t_1, t_2, ..., t_n)) = max\{rang(t_i) : 1 \leq i \leq n\} + 1$.

Assume now that $\mathcal{A} \in \mathcal{F}(A), X \subseteq A$. If $t \in X(\mathcal{F})$, the *value of t in $\mathcal{A}$* (denoted by $t^{\mathcal{A}}$) is defined by:

- If $t \in X$, then $t^{\mathcal{A}} = \{t\}$;
- If $t = f(t_1, t_2, ..., t_n)$, then $t^{\mathcal{A}} = \bigcup\{f^{\mathcal{A}}(x) : x \in t_1^{\mathcal{A}} \times t_2^{\mathcal{A}} \times ... \times t_n^{\mathcal{A}}\}$.

Thus, if $t \in \mathcal{F}_0$, then $t^{\mathcal{A}}$ is just the interpretation of the 0-ary functional symbol t in $\mathcal{A}$.

2.3. Subalgebras

Let $\mathcal{A} \in \mathcal{F}(A)$, $\mathcal{B} \in \mathcal{F}(B)$. We say that $\mathcal{B}$ is a *subalgebra* of $\mathcal{A}$ if $B \subseteq A$ and for all $n \in N, f \in \mathcal{F}_n$, $x \in B^n$ it holds $f^{\mathcal{A}}(x) = f^{\mathcal{B}}(x)$. The carrier B of the subalgebra $\mathcal{B}$ is usually called a *subobject* of $\mathcal{A}$. The least subobject B of $\mathcal{A}$ such that $X \subseteq B$ $(X \neq \emptyset)$ we denoted by $\langle X \rangle_{\mathcal{A}}$. It is not hard to see that if $\emptyset \neq X \subseteq A$, then

$$\langle X \rangle_{\mathcal{A}} = \bigcup\{t^{\mathcal{A}} : t \in X(\mathcal{F})\}.$$

If $\mathcal{A} = \langle X \rangle_{\mathcal{A}}$, then for X we say that it is a *generating set of* $\mathcal{A}$, or that X *generates* $\mathcal{A}$.

2.4. Homomorphisms

In [4] we defined three kinds of homomorphisms, each of them coincides with the usual notion of homomorphisms in the class of the usual universal algebras. Here, we will deal with two kinds of homomorphisms. In order to give definitions, let us agree that if $\varphi : A \to B$, then the induced mappings from $\mathcal{P}(A)$ into $\mathcal{P}(B)$, as well as from A^n into B^n we denote by the same symbol φ.

Definition 2.1. *Let* $\mathcal{A} \in \mathcal{F}(A), \mathcal{B} \in \mathcal{F}(B)$.

1. *A mapping* $\varphi : A \to B$ *is called a* homomorphism (from $\mathcal{A}$ to $\mathcal{B}$) *if*

   $$\varphi(f^{\mathcal{A}}(x)) \subseteq f^{\mathcal{B}}(\varphi(x)),$$

   for all $n \in N$, $f \in \mathcal{F}_n$, $x \in A^n$.

2. *A mapping* $\varphi : A \to B$ *is called a* strong homomorphism(from $\mathcal{A}$ to $\mathcal{B}$) *if*

   $$\varphi(f^{\mathcal{A}}(x)) = f^{\mathcal{B}}(\varphi(x)),$$

   for all $n \in N$, $f \in \mathcal{F}_n$, $x \in A^n$.

As usual, if f is a (strong) homomorphism from $\mathcal{A}$ to $\mathcal{B}$, then we will say " $f : \mathcal{A} \to \mathcal{B}$ is a (strong) homomorphism". Note, that in the case $f \in \mathcal{F}_0$, $f^{\mathcal{A}}(x) = f^{\mathcal{A}}$, and our conditions for φ to be a homomorphism and a strong homomorphism become

$$\varphi(f^{\mathcal{A}}) \subseteq f^{\mathcal{B}} \ , \ \varphi(f^{\mathcal{A}}) = f^{\mathcal{B}}.$$

The notion of isomorphism is defined in the usual way. Namely, if $\mathcal{A} \in \mathcal{F}(A)$, $\mathcal{B} \in \mathcal{F}(B)$, then $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be an *isomorphism* between $\mathcal{A}$ and $\mathcal{B}$ if $\varphi : A \to B$ is a bijection such that both $\varphi : \mathcal{A} \to \mathcal{B}$ and $\varphi^{-1} : \mathcal{B} \to \mathcal{A}$ are homomorphisms. We note that every bijective strong homomorphism is an isomorphism (see [4]).

2.5. Free objects

Definition 2.2. *Let $\mathcal{A} \in \mathcal{F}(A)$, $\mathcal{C} \in \mathcal{F}(C)$ and $B \subseteq A$. We say that B is a* (strong) basis of $\mathcal{A}$ over $\mathcal{C}$ *if the following conditions hold:*

1) B generates $\mathcal{A}$,

2) For every mapping $\varphi : B \to C$ there is an extension $\bar{\varphi} : A \to C$ which is a (strong) homomophism from $\mathcal{A}$ to $\mathcal{C}$.

If $\mathcal{K}$ is a class of poly-algebras, then B is a *(strong) basis of $\mathcal{A}$ over $\mathcal{K}$* if B is a (strong) basis of $\mathcal{A}$ over every object $\mathcal{C} \in \mathcal{K}$. In this case, if $\mathcal{A} \in \mathcal{K}$, we say that B is a *(strong) basis of $\mathcal{A}$ in $\mathcal{K}$*.

$\mathcal{A}$ is said to be *(strongly) free over $\mathcal{C}$* iff there is a (strong) basis B of $\mathcal{A}$ over $\mathcal{C}$. If there is a (strong) basis B of $\mathcal{A}$ over every object $\mathcal{C} \in \mathcal{K}$, then we say that $\mathcal{A}$ is *(strongly) free over $\mathcal{K}$*; if, moreover, $\mathcal{B} \in \mathcal{K}$, then $\mathcal{A}$ is a *(strongly) free object in $\mathcal{K}$*.

3. Free objects over $\mathcal{R}eg(\mathcal{F})$

In this section we shall give a complete description of all free objects over (in) the class $\mathcal{R}eg(\mathcal{F})$. As we shall see, there are non-isomorphic free objects with the same basis.

In the sequel, let us denote

$$B_n = \{t \in B(\mathcal{F}) : rang(t) \leq n\}.$$

In this way,

$$B_0 = B \cup \mathcal{F}_0,$$

$$B_{n+1} = B_n \cup \{f(t_1, ..., t_k) : k \in N \wedge f \in \mathcal{F}_k \wedge (\forall i \leq k) t_i \in B_n\},$$

for $n \in N$, and

$$B(\mathcal{F}) = \cup\{B_n : n \in N\}.$$

Lemma 3.1. *B is a basis of $B(\mathcal{F})$ in $\mathcal{R}eg(\mathcal{F})$.*

Proof.

Let $\varphi : B \to A$ and $\mathcal{A} \in \mathcal{R}eg(\mathcal{F})$. We define a sequence of mappings $\varphi_n : B_n \to A, n \in N$, in the following way:

$$\varphi_0 = \varphi,$$

$$\varphi_{n+1} |_{B_n} = \varphi_n,$$

$$\varphi_{n+1}(f(t_1, ..., t_k)) \in f^{\mathcal{A}}(\varphi_n(t_1), ..., \varphi_n(t_k)),$$

for all $f(t_1, ..., t_k) \in B_{n+1} \setminus B_n$.

If we define $\bar{\varphi} : B(\mathcal{F}) \to A$ as $\bar{\varphi} = \cup\{\varphi_n : n \in N\}$, then it is easy to see that $\bar{\varphi}$ is a homomorphism from $B(\mathcal{F})$ to $\mathcal{A}$, extending φ.

Remark 3.1. *The homomorphism $\bar{\varphi}$ defined in the proof of L3.1. is unique only in the case $\mathcal{A} \in \mathcal{U}al(\mathcal{F})$.*

Lemma 3.2. *Let B be a generating set of $\mathcal{A} \in \mathcal{F}(A)$ and there is a homomorphism $\varphi : \mathcal{A} \to B(F)$ which is an extension of the identity mapping $1 : B \to B$. Then B is a basis of $\mathcal{A}$ over $\mathcal{R}eg(\mathcal{F})$.*

Proof. It follows from L3.1 and the fact that the composition of two homomorphisms is again a homomorphism.

Lemma 3.3. *If $\mathcal{A} \in \mathcal{F}(A)$, $\emptyset \neq B \subseteq A$ and there is a homomorphism $\varphi : \mathcal{A} \to B(\mathcal{F})$ which is an extension of $1 : B \to B$, then for every $t \in B(\mathcal{F})$*

$$\varphi(t^{\mathcal{A}}) \subseteq \{t\}. \tag{1}$$

Proof. If $t \in B(\mathcal{F})$, then there exists $n \in N$ such that $t \in B_n$. The proof goes by induction on n. For $n = 0$, it is obvious. Assume that (1) is satisfied for n and let $t \in B_{n+1} \setminus B_n$. Then

$$t = f(t_1, ..., t_k), \text{ for some } f \in \mathcal{F}_k, t_1, ..., t_k \in B_n.$$

Then we have

$$\varphi(t^{\mathcal{A}}) = \varphi(f^{\mathcal{A}}(t_1^{\mathcal{A}}, ..., t_k^{\mathcal{A}})) \subseteq f^{\mathcal{B}(\mathcal{F})}(\varphi(t_1^{\mathcal{A}}), ..., \varphi(t_k^{\mathcal{A}})).$$

By the inductive assumption we have

$$\varphi(t_1^{\mathcal{A}}) \subseteq \{t_1\}, ..., \varphi(t_k^{\mathcal{A}}) \subseteq \{t_k\},$$

which implies

$$\varphi(t^{\mathcal{A}}) \subseteq \{t\}.$$

Remark 3.2. *If in L3.3. we have $\mathcal{A} \in \mathcal{R}eg(\mathcal{F})$, then*

$$(\forall t \in B(\mathcal{F}))\varphi(t^{\mathcal{A}}) = t,$$

and φ is a strong homomorphism.

Lemma 3.4. *If B is a generating subset of $\mathcal{A} \in \mathcal{F}(A)$, and if*

(2) $$u^{\mathcal{A}} \cap v^{\mathcal{A}} \neq \emptyset \;\Rightarrow\; u = v$$

for every $u, v \in B(\mathcal{F})$, then B is a basis of $\mathcal{A}$ over $\mathcal{R}eg(\mathcal{F})$.

Proof. By L3.1. we have to show that there is a homomorphism $\varphi : \mathcal{A} \to B(\mathcal{F})$ which is an extension of $1 : B \to B$.

Let $a \in A$. Because B is a generating set of $\mathcal{A}$, there is a $t \in B(\mathcal{F})$ such that $a \in t^{\mathcal{A}}$. By the condition (2), for every a, such term t is unique, so putting $\varphi(a) = t$, we obtain a desired homomorphism $\varphi : \mathcal{A} \to B(\mathcal{F})$. Note that if $\mathcal{A} \in \mathcal{R}eg(\mathcal{F})$, then φ is a strong homomorphism as well.
□

Theorem 3.1. *Let $\mathcal{A} \in \mathcal{F}(A)$ and $\emptyset \neq B \subseteq A$. Then, B is a basis of $\mathcal{A}$ over $\mathcal{R}eg(\mathcal{F})$ iff for every $a \in A$ there is a unique term $t \in B(\mathcal{F})$ such that $a \in t^{\mathcal{A}}$.*

Proof. Directly from L3.3 and L3.4.
□

Corollary 3.1. *Every free object over $\mathcal{R}eg(\mathcal{F})$ has a unique basis.*

Proof. Let $\mathcal{A} \in \mathcal{F}(A)$ be a free object over $\mathcal{R}eg(\mathcal{F})$, with the basis B. If B_1 is an other basis of $\mathcal{A}$, then $B_1 \not\subseteq B$, because B is the least generating set of $\mathcal{A}$. If $a \in B_1 \setminus B$, then there is a $t \in B(\mathcal{F}) \subseteq B_1(\mathcal{F})$ such that $t \neq a$ and $a \in t^{\mathcal{A}}$. But, in the same time, $a \in t_1^{\mathcal{A}}$, where $t_1 = a$, so $a \in t \cap t_1 \neq \emptyset$, which is in contradiction with T3.1.
□

In order to give a more explicit description of free objects over $\mathcal{R}eg(\mathcal{F})$, we shall introduce the so called *standard free objects* over (in) $\mathcal{R}eg(\mathcal{F})$.

Lemma 3.5. *Let $\emptyset \neq B \subseteq A$ and $\varphi : A \to B(\mathcal{F})$ such that*

1. *$\varphi^{-1}(b) = \{b\}$, for all $b \in B$,*
2. *If $t \in \varphi(A)$,and $t = f(t_1, t_2, ..., t_n)$, then $t_1, t_2, ..., t_n \in \varphi(A)$.*

Define a poly-$\mathcal{F}$-algebra $\mathcal{A} \in \mathcal{F}(A)$ in the following way:

$$f^{\mathcal{A}}(a) = \varphi^{-1}(f(\varphi(a))),$$

for every $n \in N$, $f \in \mathcal{F}_n$, $a \in A^n$. Then B is a basis of $\mathcal{A}$ over $\mathcal{R}eg(\mathcal{F})$.

Definition 3.1. *Let $\mathcal{A} \in \mathcal{F}(A)$ and $\emptyset \neq B \subseteq A$. Then $\mathcal{A}$ is called the* standard free object (with basis B, determined by φ) *if there is a mapping $\varphi : A \to B(\mathcal{F})$ such that*

1. *$\varphi^{-1}(b) = \{b\}$, for all $b \in B$,*
2. *If $t \in \varphi(A)$ and $t = f(t_1, t_2, ..., t_n)$, then $t_1, t_2, ..., t_n \in \varphi(A)$,*
3. *$f^{\mathcal{A}}(a) = \varphi^{-1}(f(\varphi(a)))$, for every $n \in N$, $f \in \mathcal{F}_n$, $a \in A^n$.*

Example 3.1. (*Free object which is not standard free object*) Let $\mathcal{F} = \mathcal{F}_1 = \{f\}$, $B = \{b\}$ and the poly-$\mathcal{F}$-algebra $\mathcal{A}$ is defined as follows:

$$A = \{b\} \cup \{a_n : n \geq 1\} \cup \{c_n : n \geq 1\},$$

$$f^{\mathcal{A}}(b) = \{a_1, c_1\},\ f^{\mathcal{A}}(a_n) = \{a_{n+1}\},\ f^{\mathcal{A}}(c_n) = \{c_{n+1}\},\ \text{for } n \geq 1.$$

Then it is easy to see that B is a basis of $\mathcal{A}$ in $\mathcal{R}eg(\mathcal{F})$. On the other hand, $\mathcal{A}$ is not a standard free object. Namely, if $\varphi : A \to B(\mathcal{F})$ were a mapping such that $\varphi(b) = b$, and

$$(\forall x \in A) f^{\mathcal{A}}(x) = \varphi^{-1}(f(\varphi(x))),$$

then we would have $\varphi(a_1) = \varphi(c_1) = f(b)$ and $\{a_2\} = \varphi^{-1}(f^2(b)) = \{c_2\}$, which is a contradiction.

□

Remark 3.3. *In the same way as in Example 3.1, similar examples can be constructed for any type $\mathcal{F} \neq \mathcal{F}_0$.*

Definition 3.2. *Let $\mathcal{A}, \mathcal{B} \in \mathcal{F}(A)$. Then $\mathcal{A} \leq \mathcal{B}$ if*

$$f^{\mathcal{A}}(x) \subseteq f^{\mathcal{B}}(x),$$

for all $n \in N$, $f \in \mathcal{F}_n$, $x \in A^n$.

Theorem 3.2. *Let B be a generating set of $\mathcal{A} \in \mathcal{F}(A)$. Then, B is a basis of $\mathcal{A}$ over $\mathcal{R}eg(\mathcal{F})$ iff there is a standard free object $\mathcal{C}$ with the basis B such that $\mathcal{A} \leq \mathcal{C}$.*

Proof. $(\to)$. If B is a basis of $\mathcal{A}$ over $\mathcal{R}eg(\mathcal{F})$, then there is a homomorphism $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{F})$ which is an extension of the identity mapping $1 : B \to B$. Define the poly-$\mathcal{F}$-algebra $\mathcal{C} \in \mathcal{F}(A)$ such that

$$f^{\mathcal{C}}(a) = \varphi^{-1}(f(\varphi(a))),$$

for all $n \in N$, $f \in \mathcal{F}_n$, $x \in A^n$. Then $\mathcal{C}$ will be a standard free object, such that $\mathcal{A} \leq \mathcal{C}$.

$(\leftarrow)$. It follows from L3.4.

□

Theorem 3.3. *Assume that $B \neq \emptyset$ and α is a cardinal number such that*

$$\alpha \geq max\{| B |, | \mathcal{F} |, \aleph_0\}.$$

Then there is a free object $\mathcal{A} \in \mathcal{F}(A)$ in $\mathcal{R}eg(\mathcal{F})$ with a basis B, where $| A | = \alpha$.

4. Strongly free objects in (over) $\mathcal{R}eg[\mathcal{F},\alpha]$

In [4] it is proved that the class of strongly free objects over $\mathcal{R}eg(\mathcal{F})$ is empty. Here we will show that there exist strongly free objects in the class $\mathcal{R}eg[\mathcal{F},\alpha]$ of all α-bounded regular poly-$\mathcal{F}$-algebras, for every positive cardinal α, and that all strongly free objects with a same basis B in $\mathcal{R}eg[\mathcal{F},\alpha]$ are isomorphic only in the case when α is finite or $\mathcal{F}=\mathcal{F}_0$.

In the sequel, we identify the cardinal number α with the set of all cardinals less than α.

Definition 4.1. *Let $\mathcal{F}$ be a type, α a positive cardinal number and B a non-empty set.*

1) Define the collection $\{[B,\alpha]_n : n\in N\}$ as follows:

$$[B,\alpha]_0 = B,$$

$$[B,\alpha]_{n+1} = [B,\alpha]_n \cup (\alpha \times \cup\{\mathcal{F}_k \times ([B,\alpha]_n)^k : k\in N\})$$

2) Let $[B,\alpha]=\cup\{[B,\alpha]_n : n\in N\}$. On the set $[B,\alpha]$ we define a poly-$\mathcal{F}$-algebra in the following way: for every $n\in N, f\in\mathcal{F}_n, a\in[B,\alpha]^n$, let

$$f^{\mathcal{A}}(a)=\{(\beta,f,a):\beta\in\alpha\}.$$

This algebra will be denoted by $[B,\alpha]$.

Lemma 4.1. *For every $n\in N$, the set $[B,\alpha]_{n+1}\setminus[B,\alpha]_n$ is equal to*

$$\cup\{\{(\beta,f,x):\beta\in\alpha\}:k\in N, f\in\mathcal{F}_k, x\in([B,\alpha]_n)^k\setminus([B,\alpha]_{n-1})^k\},$$

(where $[B,\alpha]_{-1}=\emptyset$), and this union is disjoint.

Lemma 4.2. *B is a strong basis of $[B,\alpha]$ in $\mathcal{R}eg[\mathcal{F},\alpha]$.*

Proof. First, it is clear that $[B,\alpha]\in\mathcal{R}eg[\mathcal{F},\alpha]$, and it can be easily shown that B is a generating set of $[B,\alpha]$. Let $\mathcal{D}\in\mathcal{F}(D)$ be an object in $\mathcal{R}eg[\mathcal{F},\alpha]$ and let $\varphi : B\to D$ be a given mapping. Define a collection of mappings $\varphi_n : [B,\alpha]_n \to D, n\in N$, as follows:
1) $\varphi_0=\varphi$,

2) If $n \geq 1$, then for every $k \in N, f \in \mathcal{F}_k, a \in ([B,\alpha]_n)^k \setminus ([B,\alpha]_{n-1})^k$ we have

$$|\{(\beta, f, a) : \beta \in \alpha\}| = \alpha \geq | f^{\mathcal{D}}(\varphi_n(a))|,$$

and thus, there is a sirjection

$$\psi_{n,f,a} : \{(\beta, f, a) : \beta \in \alpha\} \to f^{\mathcal{D}}(\varphi_n(a)).$$

Now, we define $\psi_n : [B,\alpha]_{n+1} \setminus [B,\alpha]_n$ by

$$\psi_n = \cup\{\psi_{n,f,a} : f \in \mathcal{F}_k, k \in N, a \in ([B,\alpha]_n)^k \setminus ([B,\alpha]_{n-1})^k\}.$$

Finally, we define φ_{n+1} by

$$\varphi_{n+1} = \varphi_n \cup \psi_n.$$

From L4.1. it follows that $\varphi_n, n \in N$, are well defined. Also, it is clear that for all $n \in N$,
a) φ_{n+1} is an extension of φ_n,
b) for every $k \in N, f \in \mathcal{F}_k$, $a \in ([B,\alpha]_n)^k \setminus ([B,\alpha]_{n-1})^k$

$$\varphi_{n+1}(\{(\beta, f, a) : \beta \in a\}) = f^{\mathcal{D}}(\varphi_n(a)).$$

If we define $\bar{\varphi} : [B,\alpha] \to D$ by

$$\bar{\varphi} = \cup\{\varphi_n : n \in N\},$$

then $\bar{\varphi}$ is a strong homomorphism from $[B,\alpha]$ into $\mathcal{D}$ which is an extension of φ.
□

Lemma 4.3. *If φ is a strong endomorphism of $[B,\alpha]$, such that $(\forall b \in B)\varphi(b) = b$, then the following statements hold:*
a) $\varphi(\{(\beta, f, x) : \beta \in \alpha\}) = \{(\beta, f, \varphi(x)) : \beta \in \alpha\}$, for every $n \in N$, $f \in \mathcal{F}_n, x \in [B,\alpha]^n$;
b) $\varphi([B,\alpha]_n) = [B,\alpha]_n$ for every $n \in N$;
c) φ is surjective;
d) If α is finite, then φ is an automorphism of $[B,\alpha]$.

Proof.
a) $\varphi(\{(\beta, f, x) : \beta \in \alpha\}) = \varphi(f^{[B,\alpha]}(\varphi(x))) = \{(\beta, f, \varphi(x)) : \beta \in \alpha\}$.
b) For $n = 0$, the condition is satisfied. Suppose that for all $k \leq n$ it holds

$$\varphi([B,\alpha]_k) = [B,\alpha]_k.$$

From a) and L 4.1. we obtain

$$\varphi([B,\alpha]_{n+1} \setminus [B,\alpha]) = [B,\alpha]_{n+1} \setminus [B,\alpha]_n$$

and this implies that $\varphi([B,\alpha]_{n+1}) = [B,\alpha]_{n+1}$.
Clearly, b) implies c).
d) First, because of a), for every $n \in N, f \in \mathcal{F}_n, x \in [B,\alpha]^n$,the restriction of φ to the set $\{(\beta, f, x) : \beta \in \alpha\}$ is a bijection. Assuming that the restriction of φ to $[B,\alpha]$ is a bijection, using L4.1. we obtain that φ induces a permutation of $[B,\alpha]_{n+1}$ as well. Hence φ is a bijective strong homomorphism, and therefore it is an isomorphism.
□

Corollary 4.1. *B is the unique strong basis of $[B,\alpha]$ in $\mathcal{R}eg[\mathcal{F},\alpha]$.*

Proof.
Every strong basis is a basis, so the result follows from C3.1.
□

It is now easy to give a description of strongly free objects over $\mathcal{R}eg[\mathcal{F},\alpha]$.

Theorem 4.1. *Let $\mathcal{A} \in \mathcal{F}(A)$, and B be a generating subset of $\mathcal{A}$. Then, B is a strong basis of $\mathcal{A}$ over $\mathcal{R}eg[\mathcal{F},\alpha]$ iff the following two conditions hold:*
a) For every pair $u, v \in \mathcal{B}(F)$, the following implication holds

$$u^{\mathcal{A}} \cap v^{\mathcal{A}} \neq \emptyset \;\Rightarrow\; u = v.$$

b) $(\forall n \in N)(\forall f \in \mathcal{F}_n)(\forall a \in A^n) \mid f^{\mathcal{A}}(a) \mid \geq \alpha$.

Proof.
Let B be a strong basis of $\mathcal{A}$ over $\mathcal{R}eg[\mathcal{F},\alpha]$. Then it is also a basis of $\mathcal{A}$ over $\mathcal{R}eg[\mathcal{F},1] = \mathcal{U}al(\mathcal{F})$, and thus a) holds. Let $\varphi : A \to [B,\alpha]$ be a strong homomorphism which is an extension of $1 : B \to B$. Then, we have:

$$(\forall n \in N)(\forall f \in \mathcal{F}_n)(\forall a \in A^n) \mid f^{\mathcal{A}}(a) \mid \geq \mid f^{[B,\alpha]}(\varphi(a)) \mid = \alpha,$$

i.e. b) holds.

Assume now that a) and b) hold. For every $t \in B(\mathcal{F})$, $t \notin B$, $t^{\mathcal{A}}$ is a subset of A such that $\mid t^{\mathcal{A}} \mid \geq \alpha$, and $t^{[B,\alpha]}$ a subset of $[B,\alpha]$ such that

$| t^{[B,\alpha]} |= \alpha$. Thus, for every $t \in B(\mathcal{F}) \setminus B$, there is a surjective mapping $\varphi_t : t^{\mathcal{A}} \to t^{[B,\alpha]}$; if $b \in B$, then we put $\varphi_t(b) = b$. Then

$$\varphi = \cup\{\varphi_t : t \in B(\mathcal{F})\}$$

is a (surjective) strong homomorphism from $\mathcal{A}$ into $[B, \alpha]$ which is an extension of $1 : B \to B$.

□

Theorem 4.2. *If α is finite or $\mathcal{F} = \mathcal{F}_0$, then every strongly free object $\mathcal{D} \in \mathcal{F}(D)$ with a basis B in $\mathcal{R}eg[\mathcal{F}, \alpha]$ is isomorphic to $[B, \alpha]$.*

Proof.
Let $\varphi : [B, \alpha] \to \mathcal{D}$ and $\psi : \mathcal{D} \to [B, \alpha]$ are strong homomorphisms such that

$$(\forall b \in B)\varphi(b) = \psi(b) = b.$$

Then $\varphi([B, \alpha])$ is a subobject of $\mathcal{D}$ such that $B \subseteq \varphi([B, \alpha])$ and this implies that $\varphi([B, \alpha]) = D$, i.e. φ is surjective. (Notice that the assumption that α is finite is not used.)

If α is finite then, by L4.3.d), $\Psi = \psi\varphi$ is an automorphism of $[B, \alpha]$ and this implies that φ is injective. Thus, φ is a bijective strong homomorphism and therefore (by Prop.2.11. from [4]) $\varphi : [B, \alpha] \to \mathcal{D}$ is an isomorphism.

Let $\mathcal{F} = \mathcal{F}_0$. Then

$$[B, \alpha] = B \cup \{(\beta, f) : \beta \in \alpha, f \in \mathcal{F}_0\}, \text{ and}$$

$$f^{[B,\alpha]} = \{(\beta, f) : \beta \in \alpha\}, \text{ for every } f \in \mathcal{F}_0 = \mathcal{F}.$$

Assume that $\mathcal{D} \in \mathcal{F}(D)$ is a strong free object in $\mathcal{R}eg[\mathcal{F}, \alpha]$ with a basis B, and let φ, ψ be as above. Then we have

$$f^{\mathcal{D}} = \varphi(f^{[B,\alpha]}) = \{\varphi((\beta, f)) : \beta \in \alpha\},$$

$$\{(\beta, f) : \beta \in \alpha\} = \psi(f^{\mathcal{D}}) = \{\psi\varphi((\beta, f)) : \beta \in \alpha\}.$$

From the last equalities we first obtain that $| f^{\mathcal{D}} |= \alpha$, for every $f \in \mathcal{F}$. Moreover, we have

$$B \cap f^{\mathcal{D}} = \emptyset, \text{ for every} f \in \mathcal{F},$$

$$f^{\mathcal{D}} \cap g^{\mathcal{D}} = \emptyset, \text{ for every } f, g \in \mathcal{F}, f \neq g.$$

Namely, this follows from

$$B \cap f^{[B,\alpha]} = \emptyset \text{ and}$$

$$f^{[B,\alpha]} \cap g^{[B,\alpha]} \neq \emptyset \Rightarrow f = g.$$

Thus, there exists a collection of bijections $\varphi_f : f^{[B,\alpha]} \to f^{\mathcal{D}}$, $f \in \mathcal{F}$. If we define the mapping $\eta : [B,\alpha] \to D$ by

$$\eta(b) = b, \text{ for every } b \in B,$$

$$\eta(x) = \varphi_f(x), \text{ for every } x \in f^{[B,\alpha]}, f \in \mathcal{F},$$

then η is an isomorphism from $[B,\alpha]$ onto $\mathcal{D}$.
□

Theorem 4.3. *If α is infinite and $\mathcal{F} \neq \mathcal{F}_0, B \neq \emptyset$, then there exists non-isomorphic strongly free objects in $\mathcal{R}eg[\mathcal{F},\alpha]$ with a basis B.*

Proof.
Let $1 \leq | I | \leq \alpha$ and let for $i \in I$

$$A_i = B \cup ([B,\alpha] \setminus B) \times \{i\}.$$

For any $i \in I$, define a mapping $\varphi_i : A_i \to [B,\alpha]$ such that

$$\varphi_i(b) = b, \text{ for every } b \in B,$$

$$\varphi_i((x,i)) = x, \text{ for every } x \in [B,\alpha].$$

Poly-$\mathcal{F}$-algebras $\mathcal{A}_i$ we define in the following way:

$$f^{\mathcal{A}_i}(x) = \varphi_i^{-1}(f^{[B,\alpha]}(\varphi_i(x))),$$

for every $n \in N, f \in \mathcal{F}_n, x \in A_i^n$. It is obvious that
1) $[B,\alpha] \cap A_i = A_i \cap A_j = B$, for every $i,j \in I, i \neq j$,
2) $\varphi_i : A_i \to [B,\alpha]$ are isomorphisms such that for all $i \in I$

$$(\forall b \in B)\varphi_i(b) = b.$$

Let $A = \cup\{A_i : i \in I\}, \varphi = \cup\{\varphi_i : i \in I\}$. Define a poly-algebra $\mathcal{A} \in \mathcal{F}(A)$ by:

$$f^{\mathcal{A}}(x) = \cup\{\varphi_i^{-1}(f^{[B,\alpha]}(\varphi(x))) : i \in I\},$$

for every $n \in N, f \in \mathcal{F}_n, x \in A^n$.

First of all, it is clear that $| f^{\mathcal{A}}(x) |= \alpha$, so $\mathcal{A} \in \mathcal{R}eg[\mathcal{F}, \alpha]$. Moreover, if $n \in N, f \in \mathcal{F}_n, x \in A^n$, then

$$\varphi(f^{\mathcal{A}}(x)) = \cup\{\varphi_i\varphi_i^{-1}(f^{[B,\alpha]}(\varphi(x))) : i \in I = f^{[B,\alpha]}(\varphi(x)),$$

so $\varphi : \mathcal{A} \to [B, \alpha]$ is a strong homomorphism such that $(\forall b \in B)\varphi(b) = b$.

We will show now that B is a generating subset of $\mathcal{A}$. Assume that Q is a subobject of $\mathcal{A}$ such that $B \subseteq Q$. It is not hard to see that $Q_i = Q \cap A_i$ is a subobject of $\mathcal{A}_i$, such that $B \subseteq Q_i$, for every $i \in I$. Namely, if $x \in Q_i$, then

$$f^{\mathcal{A}_i}(x) \subseteq A_i,$$

$$f^{\mathcal{A}_i}(x) = \varphi_i^{-1}\varphi_i(f^{\mathcal{A}_i}(x)) = \varphi_i^{-1}f^{[B,\alpha]}(\varphi_i(x)) = \varphi_i^{-1}f^{[B,\alpha]}(\varphi(x)) \subseteq f^{\mathcal{A}}(x) \subseteq Q,$$

so $f^{\mathcal{A}_i}(x) \subseteq Q_i$. This implies that $Q_i = A_i$, for every $i \in I$, and thus $Q = A$, i.e. B is a generating subset of $\mathcal{A}$.

Let $\mathcal{D} \in \mathcal{F}(D) \cap \mathcal{R}eg[\mathcal{F}, \alpha]$ and let $\psi : B \to D$. Then there is a strong homomorphism $\bar{\psi} : [B, \alpha] \to \mathcal{D}$ which is an extension of ψ, and therefore $\bar{\psi}\varphi : \mathcal{A} \to \mathcal{D}$ is also a strong homomorphism which is an extension of ψ.

This complete the proof that B is a strong basis of $\mathcal{A}$ in $\mathcal{R}eg[\mathcal{F}, \alpha]$.

Assume now that $| I |> 1$ and let $i, j \in I, i \neq j$. Let $x \in [B, \alpha]^n \setminus B^n, f \in \mathcal{F}_n$, and

$$y = \varphi_i^{-1}(x),\ z = \varphi_j^{-1}(x).$$

Then we have $y \neq z$, and

$$f^{\mathcal{A}}(y) = \cup\{\varphi_k^{-1}(f^{[B,\alpha]}(\varphi(y))) : k \in I\} = \cup\{\varphi_k^{-1}(f^{[B,\alpha]}(x)) : k \in I\} = f^{\mathcal{A}}(z).$$

This implies that $\mathcal{A}$ and $[B, \alpha]$ are not isomorphic, for if $\Psi : \mathcal{A} \to [B, \alpha]$ were an isomorphism, then we would have

$$f^{[B,\alpha]}(\Psi(y)) = f^{[B,\alpha]}(\Psi(z)),$$

and $\Psi(y) \neq \Psi(z)$, which is impossible.
□

Remark 4.1. *The description of free objects in $\mathcal{R}eg[\mathcal{F}, \alpha]$ is given in the previous section. We note that if $1 \leq \beta < \alpha$, then $[B, \alpha]$ is a free object in $\mathcal{R}eg[\mathcal{F}, \alpha]$, but $[B, \beta]$ is not a strongly free object in $\mathcal{R}eg[\mathcal{F}, \alpha]$. Clearly, every strongly free object in $\mathcal{R}eg[\mathcal{F}, \alpha]$ is a free object in $\mathcal{R}eg[\mathcal{F}, \alpha]$, as well.*

References

[1] Bruck, R.H., A Survey of Binary Systems, Springer-Verlag, Berlin-Gottingen- Heidelberg, 1958.

[2] Comer,S.D., A new foundation for the theory of relations, Preprint, 1981.

[3] Comer, S.D., Combinatorial aspects of relations, Algebra Universalis, 18(1984)77-94.

[4] Čupona, Ǵ., Madarász, R., On poly-algebras, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 21,2(1991)

[5] Čupona,G., Markovski, S., Free Objects in the Class of Vector Valued Grupoids Induced by Semigroups , Contributions to General Algebra 7, Wien, 1991, 87-95.

[6] Dresher,M., Ore, M., Theora of multigroups, Amer. J. Math. 60 (1938), 705-733.

[7] Goldblatt, R., Varieties of Complex Algebras , Annals of Pure and Applied Logic 44(1989), 173-242

[8] Grätzer,G., Universal Algebra, Second Ed., Springer-Verlag, New York-Heidelberg-Berlin, 1979.

[9] Jónsson, B., The theory of binary relations,in: Algebraic Logic, Coll. Math. Soc. J. Bolyai, 54, North Holland,1988, 245-292.

[10] Jónsson,B., Tarski, A., Boolean algebras with operators II, Amer.J. Math. 74 (1952) 127-162.

REZIME

O SLOBODNIM POLI - ALGEBRAMA

U ovom radu su posmatrane dve vrste slobodnih struktura: "slobodne" i "jako slobodne" poli-algebre. Dat je kompletan opis slobodnih objekata nad klasom regularnih poli-algebri i pokazana egzistencija jako slobodnih objekata u klasi α- ograničenih regularnih objekata, gde je α nenula kardinal.

Ako je α konačan, dva jako slobodna objekta sa istom bazom su izomorfna, i u tom slučaju imamo kompletan opis jako slobodnih objekata. Ali ako je α beskonačan, onda postoje neizomorfni slobodni objekti sa istom bazom.

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