Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 23, 2 (1993), 263 - 271

Review of Research
Faculty of Science
Mathematics Series

REDUCIBILITY OF H-COMMUTATIVE n-QUASIGROUPS

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Abstract

An n-quasigroup (Q, f) is reducible if it can be represented by two quasigroups of smaller arities, and it is completely reducible if it can be represented by $n-1$ binary quasigroups. It is proved that if an A_n-commutative n-quasigroup is completely reducible, where A_n is the alternating subgroup of the symmetrtic group S_n of degree n, then there exists an abelian group to which all component binary quasigroups are isotopic. It is also proved that for every subgroup $H \subseteq S_n$ there exists a nonreducible exactly H-commutative n-quasigroup of every composite order mp, where $m > 2$, $p \geq n \geq 3$.

AMS Mathematics Subject Classification (1991): 20N15
Key words and phrases: n-quasigroup, isotopy, reducibility, H-commutative.

1. Definitions and Notations

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_m, x_{m+1}, \ldots, x_n$ we shall denote by x_m^n or $\{x_i\}_{i=m}^n$. If $m > n$, then x_m^n will be considered empty. The sequence $x, x, \ldots, x$ (m times) will be denoted by $\overset{m}{x}$. If $m \leq 0$, then $\overset{m}{x}$ will be considered empty.

An n-ary groupoid (n-groupoid) (Q, f) is called an n-quasigroup if the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in \{1, \ldots, n\} = N_n$.

An n-quasigroup (Q, f) is isotopic to an n-quasigroup (Q, g) if there exists a sequence $T = (\alpha_1^{n+1})$ of permutations of Q such that the following identity

$$g(x_1^n) = \alpha_{n+1}^{-1} f(\{\alpha_i x_i\}_{i=1}^n)$$

holds.

By S_n we denote the symmetric group of degree n, by A_n its alternating subgroup.

An n-quasigroup (Q, f) such that

$$(1) \qquad f(x_1^n) = x_{n+1} \iff f(\{x_{\sigma(i)}\}_{i=1}^n) = x_{\sigma(n+1)}$$

for all $\sigma \in S_{n+1}$ is called totally symmetric (TS). If (1) is valid for all $\sigma \in A_{n+1}$, then f is alternating symmetric (AS).

An n-quasigroup (Q, f) is called σ-commutative if for all $x_1^n \in Q$

$$f(x_1^n) = f(\{x_{\sigma(i)}\}_{i=1}^n),$$

where σ is a permutation from S_n. If f is σ-commutative for every $\sigma \in H \subseteq S_n$, then we say that f is H-commutative. The set of all permutations σ such that (Q, f) is σ-commutative is a group.

If $H \subseteq S_n$ is the set of all permutations σ such that f is σ-commutative, then f is called exactly H-commutative.

2. Reducible A_n-commutative n-quasigroups

Definition 1. *An n-quasigroup (Q, f), $n \geq 3$, is called reducible if there exist a p-quasigroup (Q, g) and a q-quasigroup (Q, h) such that for some $i \in N_n$ and all $x_1^n \in Q$*

$$f(x_1^n) = g(x_1^{i-1}, h(x_i^{i+q-1}), x_{i+q}^n).$$

We shall introduce the following notation. For example, if g_1^5 are binary operations on a set Q and if

$$w = g_1(g_2(x_1^2), g_3(g_4(x_3, g_5(x_4^5)), x_6)), \tag{2}$$

then (2) we shall write as

$$w = g_1 g_2 x_1^2 g_3 g_4 x_3 g_5 x_4^6,$$

or

$$w = g_1^1 g_2^1 g_3^3 g_4^3 g_5^4 x_1^6,$$

where each of the upper indices of operations g_i denotes the index of the first free variable which comes after g_i.

Definition 2. *An n-quasigroup (Q, f), $n \geq 3$, is completely reducible if there exist binary quasigroups $(Q, g_1), \ldots, (Q, g_{n-1})$ such that for some $k_1^{n-1} \in N_{n-1}$ and all $x_1^n \in Q$*

$$f(x_1^n) = \{g_i^{k_i}\}_{i=1}^{n-1} x_1^n,$$

where k_i denotes the index of the first free variable which comes after g_i.

The numbers k_1^{n-1} from the preceding definition obviously satisfy $k_1 = 1$, $k_i \leq k_{i+1}$, $k_i \leq i$. Besides the numbers k_1^{n-1} we also define the sequence l_1^{n-1}, where l_i is the index of the last free variable inside the brackets which delimit the operation g_i. So, in our example $(k_1^5) = (1, 1, 3, 3, 4)$, $(l_1^5) = (6, 2, 6, 5, 5)$.

Theorem 1. *If an A_n-commutative n-quasigroup (Q, f), $n \geq 3$, is completely reducible,*

$$f(x_1^n) = \{g_i^{k_i}\}_{i=1}^{n-1} x_1^n, \tag{3}$$

then there exists an abelian group $(Q, +)$ such that each of the quasigroups g_i, $i \in N_{n-1}$, is isotopic to $(+)$.

Proof. If we choose any $t \in \{2, \ldots, n-1\}$, $a \in Q$, and if $k_t \neq 1$, then we put in (3) $x_i = a$ for all $i \in N_n \setminus \{1, k_t, l_t\}$ and get

$$f(x_1, \overset{k_t-2}{a}, x_{k_t}, \overset{l_t-k_t-1}{a}, x_{l_t}, \overset{n-l_t}{a}) = g_1(\alpha g_t(\beta x_1, \gamma x_{k_t}), \delta x_{l_t}) \tag{4}$$

or

$$f(x_1, \overset{k_t-2}{a}, x_{k_t}, \overset{l_t-k_t-1}{a}, x_{l_t}, \overset{n-l_t}{a}) = g_1(\alpha' x_1, \beta' g_t(\gamma' x_{k_t}, \delta' x_{l_t})) \tag{5}$$

where $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ are permutations of the set Q.

If $k_t = 1$, then we fix in (3) all elements x_i except x_1, x_{l_t}, x_n and get an equation similar to (4). We shall consider equation (4), the other case (5) is analogous.

We introduce two isotopes of the quasigroups g_1 and g_t by

$$h_1(x, y) = g_1(\alpha x, \delta y),$$

$$h_2(x, y) = g_t(\beta x, \gamma y).$$

To simplify the notation we shall write $x_1 = x, \quad x_{k_t} = y, \quad x_{l_t} = z, \quad k_t = p, \quad l_t = q$, and so (4) becomes

$$f(x, \overset{p-2}{a}, y, \overset{q-p-1}{a}, z, \overset{n-q}{a}) = h_1(h_2(x, y), z). \tag{6}$$

Since f is A_n-commutative,

$$f(x, \overset{p-2}{a}, y, \overset{q-p-1}{a}, z, \overset{n-q}{a}) = f(y, \overset{p-2}{a}, z, \overset{q-p-1}{a}, x, \overset{n-q}{a}),$$

hence for all $x, y, z \in Q$

$$h_1(h_2(x, y), z) = h_1(h_2(y, z), x). \tag{7}$$

If we put $x = z = a$ in (7), we have

$$R_1 L_2 y = R_1 R_2 y, \tag{8}$$

where the permutations R_i, L_i, $i = 1, 2$ of the set Q are defined by $R_i x = h_i(x, a)$, $L_i x = h_i(a, x)$, $i = 1, 2$. From (8) it follows that $L_2 = R_2$ for every a, hence h_2 is commutative.

Now by substitution $x = a$ in (7) we get

$$h_1(L_2 y, z) = R_1 h_2(y, z),$$

that is, $h_1(y, z) = R_1 h_2(L_2^{-1} y, z)$.

Let A be an isotope of h_1 defined by

$$h_1(x, y) = R_1 A(x, R_2 y). \tag{9}$$

Then

$$h_2(x,y) = A(R_2x, R_2y). \tag{10}$$

From (10) since h_2 is commutative, it follows that A is commutative.

Substituting h_1 and h_2 by A in (7), we get

$$R_1A(A(R_2x, R_2y), R_2z) = R_1A(A(R_2y, R_2z), R_2x),$$

that is

$$A(A(x,y),z) = A(A(y,z),x),$$

and by the commutativity of A

$$A(A(x,y),z) = A(x, A(y,z)),$$

so A is associative, hence A is an abelian group.

Since isotopy is an equivalence relation on the set of all quasigroups defined on Q, we get that A, g_1 and g_t are isotopic.

Choosing another $t_0 \neq t$, we get analogously that g_1 and g_{t_0} are isotopic, which implies that all quasigroups g_1^{n-1} are isotopic to the abelian group A.

□

The preceding theorem can be applied to two important classes of n-quasigroups – TS and AS n-quasigroups ([5],[6]).

Corollary 1. *If TS (AS) n-quasigroup (Q,f) is completely reducible $f(x_1^n) = \{g_i^{k_i}\}_{i=1}^{n-1} x_1^n$, then there exists an abelian group $(Q,+)$ such that each of the quasigroups g_1^{n-1} is isotopic to $(+)$.*

Remark. In [4] (Lemma 4, p.181) it was erroneously stated that if a TS n-quasigroup (Q,f) is completely reducible, $f(x_1^n) = \{g_i^{k_i}\}_{i=1}^{n-1} x_1^n$, then $g_1 = \ldots = g_{n-1}$. A counter example for this statement was given in [7].

3. The existence of nonreducible exactly H-commutative n-quasigroups

Now we shall prove that for every $H \subseteq S_n$ there exist nonreducible exactly H-commutative n-quasigroups of composit orders. In order to prove this we shall use a theorem of Belousov and Sandik [2] which states that an n-quasigroup is reducible iff it satisfies the condition D_{ij} for some $i, j \in N_n$.

An n-quasigroup (Q, f) satisfies the condition D_{ij}, where $1 \leq i < j \leq n$, if the equality

$$f(x_1^{i-1}, u_i^j, x_{j+1}^n) = f(x_1^{i-1}, v_i^j, x_{j+1}^n)$$

for some $x_1^{i-1}, u_i^j, x_{j+1}^n, v_i^j \in Q$, $(u_i^j) \neq (v_i^j)$, implies that

$$f(y_1^{i-1}, u_i^j, y_{j+1}^n) = f(y_1^{i-1}, v_i^j, y_{j+1}^n)$$

for all y_1^{i-1}, $y_{j+1}^n \in Q$.

Theorem 2. *For all positive integers m, n, p such that $n \geq 3$, $p \geq n$, $m > 2$, and every subgroup $H \subseteq S_n$, there exists nonreducible exactly H-commutative n-quasigroup of order mp.*

Proof. The construction used in this proof is based on the modified ω-wreath product from [3].

Let Z_k be the ring of integers modulo k. We define a mapping $\phi : Z_p \times Z_m \to Z_m$ by

$$\phi(x, y) = \begin{cases} 1, & x \in \{0, 1\}, \\ y, & x \notin \{0, 1\}. \end{cases}$$

Then we define an n-quasigroup (Z_p, f) by

$$f(x_1^n) = \sum_{k=1}^{n} x_k + s,$$

where s is a fixed element from Z_p, and a mapping $\theta : Z_p^n \to Z_m$ by

$$\theta(x_1^n) = \prod_{k=1}^{n} \phi(x_k, a),$$

where $a \neq 1$ is a fixed invertible element from Z_m. Further, for every $(x_1^n) \in Z_p^n$ we define a mapping $\omega_{(x_1^n)} : Z_m^n \to Z_m$ by

$$\omega_{(x_1^n)}(y_1^n) = \theta(x_1^n)(\sum_{k=1}^{n} y_k + c_{(x_1^n)}),$$

where $c_{(x_1^n)}$ is an element from Z_m which depends on (x_1^n).

Let H be any subgroup of S_n and choose n distinct elements $a_1^n \in Z_p$, two distinct elements $b, c \in Z_m$ and define an n-quasigroup $(Z_p \times Z_m, F)$ by

$$F(\{(x_i, y_i)\}_{i=1}^n) = (f(x_1^n), \omega_{(x_1^n)}(y_1^n)),$$

where $c_{(x_1^n)} = b$ if for some $\sigma \in H$ $(x_1^n) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ and $c_{(x_1^n)} = c$ otherwise. It is not difficult to see that F is an exactly H-commutative n-quasigroup.

We shall prove that F does not satisfy the conditions $D_{i,j}$, that is, that F is not reducible.

Let

$$\begin{aligned} &x_k = (1,0), \;\; k = 2, \ldots, i-1, j+2, \ldots, n,\\ &u_k = (0,0), \;\; k = i, \ldots, j,\\ &v_k = (0,0), \;\; k = i+2, \ldots, j,\\ &v_i = (1,0), \;\; v_{i+1} = (p-1, 0),\\ &x_1 = (1,-c), \;\; x_{j+1} = (1,0), \quad \text{if } i > 1,\\ &x_{j+1} = (1,-c), \quad \text{if } i = 1. \end{aligned}$$

Then it is easy to check that

$$F(x_1^{i-1}, u_i^j, x_{j+1}^n) = F(x_1^{i-1}, v_i^j, x_{j+1}^n), \qquad (u_i^j) \neq (v_i^j).$$

Assume now that for all $y_1^{i-1}, y_{j+1}^n \in Z_p \times Z_m$

$$F(y_1^{i-1}, u_i^j, y_{j+1}^n) = F(y_1^{i-1}, v_i^j, y_{j+1}^n). \tag{11}$$

If $x \in Z_p \times Z_m$, then by $x^{(1)}$ and $x^{(2)}$ we denote the components of x, that is, $x = (x^{(1)}, x^{(2)})$. (11) is equivalent to the next two equalities

$$\sum_{k=1}^{i-1} y_k^{(1)} + \sum_{k=i}^{j} u_k^{(1)} + \sum_{k=j+1}^{n} y_k^{(1)} + s = \sum_{k=1}^{i-1} y_k^{(1)} + \sum_{k=i}^{j} v_k^{(1)} + \sum_{k=j+1}^{n} y_k^{(1)} + s$$

which is obviously true, and

$$\begin{aligned} \theta(A^{(1)}) \left(\sum_{k=1}^{i-1} y_k^{(2)} + \sum_{k=i}^{j} u_k^{(2)} + \sum_{k=j+1}^{n} y_k^{(2)} + c_{A^{(1)}} \right) &= \\ \theta(B^{(1)}) \left(\sum_{k=1}^{i-1} y_k^{(2)} + \sum_{k=i}^{j} v_k^{(2)} + \sum_{k=j+1}^{n} y_k^{(2)} + c_{B^{(1)}} \right) & \end{aligned} \tag{12}$$

where $A = (y_1^{i-1}, u_i^j, y_{j+1}^n)$, $B = (y_1^{i-1}, v_i^j, y_{j+1}^n)$ and $c_{A^{(1)}} = c_{B^{(1)}} = c$ since $u_i = u_{i+1}$. From

$$\sum_{k=i}^{j} u_k^{(2)} = \sum_{k=i}^{j} v_k^{(2)} = 0$$

and (12) which is valid for all $\{y_k^{(1)}\}_{k=1}^{i-1}, \{y_k^{(1)}\}_{k=j+1}^{n}$ it follows that $\theta(A^{(1)}) = \theta(B^{(1)})$, that is,

$$\prod_{k=1}^{i-1} \phi(y_k^{(1)}, a) \prod_{k=i}^{j} \phi(u_k^{(1)}, a) \prod_{k=j+1}^{n} \phi(y_k^{(1)}, a) = \prod_{k=1}^{i-1} \phi(y_k^{(1)}, a) \prod_{k=i}^{j} \phi(v_k^{(1)}, a) \prod_{k=j+1}^{n} \phi(y_k^{(1)}, a),$$

consequently

$$\prod_{k=i}^{j} \phi(u_k^{(1)}, a) = \prod_{k=i}^{j} \phi(v_k^{(1)}, a).$$

But

$$\prod_{k=i}^{j} \phi(u_k^{(1)}, a) = 1, \qquad \prod_{k=i}^{j} \phi(v_k^{(1)}, a) = a,$$

which is a contradiction.

So, the exactly H-commutative n-quasigroup F does not satisfy the conditions D_{ij}, which means that F is not reducible. □

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REZIME

SVODLJIVOST H-KOMUTATIVNIH n-KVAZIGRUPA

n-kvazigrupa je svodljiva ako se može prikazati pomoću dve kvazigrupe manjih arnosti, a potpuno je svodljiva ako se može prikazati pomoću $n-1$ binarne kvazigrupe. Ako je A_n-komutativna n-kvazigrupa potpuno svodljiva, gde je A_n alternativna podgrupa simetrične grupe S_n stepena n, onda je dokazano da postoji abelova grupa koja je izotopna svim komponentnim binarnim kvazigrupama. Takodje je dokazano da za svaku podgrupu $H \subseteq S_n$ postoji nesvodljiva tačno H-komutativna n-kvazigrupa reda mp, gde je $m > 2$, $p \geq n \geq 3$.

Received by the editors October 13, 1993