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ON BEST APPROXIMATIONS FOR MULTIVALUED MAPPINGS IN PSEUDOCONVEX METRIC SPACES

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Abstract

We prove a generalization of the Ky Fan [1] best approximations theorem for multivalued mappings in pseudoconvex metric spaces.

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1. Introduction

Best approximations theorems for multivalued mappings are proved in [4] and [5] in locally convex Hausdorff topological vector spaces . It is well known that KKM theory is very useful in the fixed point theory and in the best approximations theory. Using a generalization of the KKM principle, proved by Ch. Horvath [3], we shall prove a best approximations theorem for multivalued mappings in pseudoconvex metric spaces. As an application a theorem on the approximate fixed point for multivalued mappings is proved.

2. Preliminaries

In [3] the following definition is introduced.

Definition 1. *Let X be a topological space and $h : X \times X \times [0,1] \to X$ so that:*

(i) $h(x,y,0) = y$, $h(x,y,1) = x$, for every $(x,y) \in X \times X$.

(ii) For every finite subset $A \subset X$, $h|co_h(A) \times co_h(A) \times [0,1]$ is continuous, where $co_h(A)$ is the convex hull of A with respect to h.

Then h is a pseudoconvex structure on X and (X,h) a pseudoconvex space.

Let (X,h) be a pseudoconvex space and $R : X \to 2^X$ (the family of all nonempty subsets of X). The mapping R is said to be an element of $KKM_h(X)$ [3] if for every finite subset $A \subset X$:

$$co_h(A) \subseteq \bigcup_{x \in A} R(X).$$

In [3] the following theorem is proved.

Theorem A. *Let (X,d,h) be a complete pseudometric space and $R \in KKM_h(X)$ such that $R(x)$ is closed for every $x \in X$. If for every $\varepsilon > 0$ there exists a finite set A such that $\alpha(\bigcap_{x \in A} R(x)) < \varepsilon$, where α is the Kuratowski measure of noncompactness, then*

$$M = \bigcap_{x \in X} R(x) \neq \emptyset$$

and M is compact.

Every normed space is a pseudoconvex space, if $h(x,y,\lambda) = \lambda x + (1-\lambda y)$. In 1970. [6] Takahashi introduced the notion of a metric space with a convex structure.

Definition 2. *Let (X,d) be a metric space and $W : X \times X \times [0,1] \to X$. The mapping W is a convex structure on X if for all $x, y \in X$ and $\lambda \in [0,1]$*

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda) d(u,y)$$

for every $u \in X$. Then (X,d,W) is a convex metric space.

If W is continuous or $co_W(A)$ is compact for every finite $A \subset X$ then (X, d, W) is a pseudoconvex metric space.

Talman introduced in [7] the notion of a strongly convex metric space in the following way.

Definition 3. *Let (X, d) be a metric space and*

$$P = \{(t_1, t_2, t_3) \in [0,1] \times [0,1] \times [0,1], \ \ t_1 + t_2 + t_3 = 1\}.$$

A strongly convex structure (SCS) on X is a continuous function $K : X \times X \times X \times P \to X$ with the property that for each $(x_1, x_2, x_3, t_1, t_2, t_3) \in X \times X \times X \times P$, $K(x_1, x_2, x_3, t_1, t_2, t_3)$ is the unique point of X which satisfies

$$d(y, K(x_1, x_2, x_3, t_1, t_2, t_3)) \leq \sum_{k=1}^{3} t_k d(y, x_k),$$

for every $y \in X$.

If (X, d, W) is strongly convex metric space and K its SCS then $W_K : X \times X \times [0,1] \to X$, defined by:

$$W_K(x_1, x_2, t) = K(x_1, x_2, x_1, t, 1-t, 0)$$

is a Takahashi convex structure.

If (X, d, W) is strongly convex metric space $co_W(A)$ is compact for every finite A.

3. A theorem on best approximations

Definition 4. *Let (X, d, h) be a pseudoconvex metric space, M a nonempty convex subset of X and $g : M \to X$. The mapping g is said to be generalized h-almost affine if the following condition (a) is satisfied:*

For every compact and convex subset $A \subset X$, every $n \in \mathbf{N}$, every $\{z_1, z_2, \ldots, z_n\} \subset A$ and every $\{x_1, x_2, \ldots x_n\} \subset M$:

$$\text{(a)} \qquad \min_{z \in A} d(g(y), z) \leq \max_{1 \leq i \leq n} d(g(x_i), z_i),$$

where y is an arbitrary element from $co_h\{x_1, x_2, \ldots x_n\}$.

Remark. If $A = \{z\}$, (a) reduces to the condition:

$$d(g(y), z) \leq \max_{1 \leq i \leq n} d(g(x_i), z)$$

i.e. g is also an h-almost affine mapping [2].

Lemma 1. *If $(X, ||\cdot||)$ is a normed space, M a nonempty, convex subset of X, and $g : M \to X$ such that (b) holds:*

(b) For every $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, every $x_1, x_2 \in M$ and every $z_1, z_2 \in X$:
$||g(\lambda_1 x_1 + \lambda_2 x_2) - \lambda_1 z_1 - \lambda_2 z_2|| \leq \max_{1 \leq i \leq 2} d(g(x_i), z_i)$

then (a) holds for $h(x, y, \lambda) = \lambda x + (1 - \lambda) y$ $(x, y \in X;\ \lambda \in [0, 1])$.

Proof. By induction in $n \in \mathbf{N}$ we shall prove that (b) implies (c):

(c) For every $\lambda_1, \lambda_2, \ldots \lambda_n \geq 0$, $\lambda_1 + \lambda_2 + \cdots \lambda_n = 1$, and every $(x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n) \in M^n \times X^n$:
$||g(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i z_i|| \leq \max_{1 \leq i \leq n} d(g(x_i), z_i)$

Indeed, suppose that (c) holds for $n = m$ and prove (c) for $n = m + 1$. We have that:

$$||g(\sum_{i=1}^{m+1} \lambda_i x_i) - \sum_{i=1}^{m+1} \lambda_i z_i|| = ||g[(1 - \lambda_{m+1})(\sum_{i=1}^{m} \frac{\lambda_i}{1 - \lambda_{m+1}} x_i) + \lambda_{m+1} x_{m+1}]$$

$$-[(1 - \lambda_{m+1})(\sum_{i=1}^{m} \frac{\lambda_i}{1 - \lambda_{m+1}} z_i) + \lambda_{m+1} z_{m+1}]||.$$

Since M is convex and $\sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_{m+1}} = 1$ it follows that $\sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_{m+1}} x_i = x \in M$ and if $z = \sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_{m+1}} z_i$ we have that (b) implies:

$$||g[(1 - \lambda_{m+1})x + \lambda_{m+1} x_{m+1}] - [(1 - \lambda_{m+1})z + \lambda_{m+1} z_{m+1}]||$$

$$\leq \max\{||g(x) - z||, ||g(x_{m+1}) - z_{m+1}||\}.$$

Since (c) holds for $n = m$ we have that

$$||g(x) - z|| = ||g(\sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_{m+1}} x_i) - \sum_{i=1}^{m} \frac{\lambda_i}{1-\lambda_{m+1}} z_i|| \leq$$

$$\leq \max_{1\leq i\leq m} ||g(x_i) - z_i||$$

and so:

$$||g(\sum_{i=1}^{m+1} \lambda_i x_i) - \sum_{i=1}^{m+1} \lambda_i z_i|| \leq \max_{1\leq i\leq m+1} ||g(x_i) - z_i||.$$

Suppose now that A is a compact and convex subset of X, $\{z_1, z_2, \ldots, z_n\} \subset A$, $\{x_1, x_2, \ldots, x_n\} \subset M$ and $y = \sum_{i=1}^{n} \lambda_i x_i$. Then

$$\min_{z\in A} ||g(y) - z|| \leq ||g(\sum_{i=1}^{n} \lambda_i x_i) - \sum_{i=1}^{n} \lambda_i z_i|| \leq$$

$$\leq \max_{1\leq i\leq n} ||g(x_i) - z_i||$$

since $\sum_{i=1}^{n} \lambda_i z_i \in A$.

Lemma 2. *If (X, d, h) is a convex metric space, where $h = W$ satisfies (d):*

(d) $d(W(x_1, x_2, \lambda), W(z_1, z_2, \lambda)) \leq \lambda d(x_1, z_1) + (1-\lambda) d(x_2, z_2)$, *for every* $x_i, z_i \in X$ $(i \in \{1, 2\}), \lambda \in [0, 1]$

then (a) holds for $g(x) = x$, for every $x \in X$.

Proof. Let A be a compact and convex subset of X. It is known that for every $B \subset X$:

$$coB = \bigcup_{n\in\mathbf{N}} \tilde{W}^n(B)$$

where $\tilde{W}^n(B) = W(\tilde{W}^{n-1}(B))$, $n \geq 2$,

$$\tilde{W}^1(B) = \{W(x, y; \lambda);\ \lambda \in [0, 1];\ x, y \in B\}.$$

It is easy to see that for $B = \{x_1, x_2, \ldots, x_m\}$, $z \in \tilde{W}^n(B)$ if and only if z is of the form:

$$z = \bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1}),$$

for some $\lambda_i \in [0,1]$ $(i \in \{1, 2, \ldots, 2^n - 1\})$, where $\bar{x}_i \in B$ $(i \in \{1, 2, \ldots, 2^n\})$ and $\bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1})$ is defined by:

$$\bar{W}(\bar{x}_1, \bar{x}_2, \lambda) = W(\bar{x}_1, \bar{x}_2, \lambda)$$

$$\begin{aligned}\bar{W}(\bar{x}_1, \bar{x}_2, & \ldots, \bar{x}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1}) = \\ & \bar{W}(\bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^{n-1}}; \lambda_1, \lambda_2, \ldots, \lambda_{2^{n-1}-1}), \\ & \bar{W}(\bar{x}_{2^{n-1}+1}, \bar{x}_{2^{n-1}+2}, \ldots, \bar{x}_{2^n}; \lambda_{2^{n-1}}, \lambda_{2^{n-1}+1}, \ldots, \lambda_{2^n-2}), \lambda_{2^n-1}).\end{aligned}$$

We shall prove that for every $\{x_1, x_2, \ldots, x_m\} \subset X$, every $y \in co_W\{x_1, x_2, \ldots, x_m\}$ and $\{z_1, z_2, \ldots, z_m\} \subset A$:

$$\min_{z \in A} d(y, z) \leq \max_{1 \leq i \leq m} d(x_i, z_i).$$

If $y \in co_W\{x_1, x_2, \ldots, x_m\}$ then

$$y \in \tilde{W}^n(\{x_1, x_2, \ldots, x_m\})$$

for some $n \in \mathbf{N}$, which means that

$$y = \bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1}),$$

$\bar{x}_i \in \{x_1, x_2, \ldots, x_m\}$ $(i \in \{1, 2, \ldots, 2^n\})$ and $\lambda_i \geq 0$ $(i \in \{1, 2, \ldots, 2^n - 1\})$.

Let $z = \bar{W}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1})$, where $\bar{z}_i = z_{k(i)} \in \{z_1, z_2, \ldots, z_m\}$ if and only if $\bar{x}_i = x_{k(i)} \in \{x_1, x_2, \ldots, x_m\}$.

We shall prove that (d) implies:

$$\begin{aligned}d(& \bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1}), \\ & \bar{W}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1})) \leq \max_{1 \leq i \leq m} d(x_i, z_i).\end{aligned} \tag{1}$$

From (d) it follows that:

$$d(\bar{W}(\bar{x}_1, \bar{x}_2, \lambda_1), \bar{W}(\bar{z}_1, \bar{z}_2, \lambda_1)) = d(W(\bar{x}_1, \bar{x}_2, \lambda_1), W(\bar{z}_1, \bar{z}_2, \lambda_1))$$

$$\leq \lambda_1 d(\bar{x}_1, \bar{z}_1) + (1 - \lambda_1) d(\bar{x}_2, \bar{z}_2) \leq \max_{1 \geq i \leq m} d(x_i, z_i).$$

Suppose that (1) holds for $n = k$ and prove (1) for $n = k + 1$.

We have that

$$d\big(\bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^{k+1}}; \lambda_1, \lambda_2, \ldots, \lambda_{2^{k+1}-1}), \bar{W}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{2^{k+1}}; \lambda_1, \lambda_2, \ldots, \lambda_{2^{k+1}-1})\big) =$$

$$= d\Big(\bar{W}\big(\bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^k}; \lambda_1, \lambda_2, \ldots, \lambda_{2^k-1}), \bar{W}(\bar{x}_{2^k+1}, \ldots, \bar{x}_{2^{k+1}}; \lambda_{2^k}, \ldots, \lambda_{2^{k+1}-2}), \lambda_{2^{k+1}-1}\big), \bar{W}\big(\bar{W}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{2^k}; \lambda_1, \lambda_2, \ldots, \lambda_{2^k-1}), \bar{W}(\bar{z}_{2^k+1}, \ldots, \bar{z}_{2^{k+1}}; \lambda_{2^k}, \ldots, \lambda_{2^{k+1}-2}), \lambda_{2^{k+1}-1}\big)\Big) \leq$$

$$\leq \lambda_{2^{k+1}-1} d\Big(\bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^k}; \lambda_1, \lambda_2, \ldots, \lambda_{2^k-1}), \bar{W}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{2^k}; \lambda_1, \lambda_2, \ldots, \lambda_{2^k-1})\Big) +$$

$$+(1 - \lambda_{2^{k+1}-1}) d\Big(\bar{W}(\bar{x}_{2^k+1}, \ldots, \bar{x}_{2^{k+1}}; \lambda_{2^k}, \ldots, \lambda_{2^{k+1}-2}), \bar{W}(\bar{z}_{2^k+1}, \ldots, \bar{z}_{2^{k+1}}; \lambda_{2^k}, \ldots, \lambda_{2^{k+1}-2})\Big)$$

$$\leq \lambda_{2^{k+1}-1} \max_{1 \leq i \leq m} d(x_i, z_i) + (1 - \lambda_{2^{k+1}-1}) \max_{1 \leq i \leq m} d(x_i, z_i) =$$

$$= \max_{1 \leq i \leq m} d(x_i, z_i).$$

From (1) we have that:

$$\min_{u \in A} d(y, u) \leq d(y, z) = d(\bar{W}(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1}),$$

$$\bar{W}(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1}))$$

since A is convex and $z = W(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{2^n}; \lambda_1, \lambda_2, \ldots, \lambda_{2^n-1}) \in coA$. Hence (1) implies that

$$\min_{u \in A} d(y, u) \leq \max_{1 \leq i \leq m} d(x_i, z_i).$$

In the next theorem $\mathcal{K}(X)$ is the family of all nonempty, convex and compact subsets of X.

Theorem 1. *Let (X, d, h) be a pseudoconvex metric space, $\emptyset \neq M$ a convex and complete subset of X, g a continuous generalized h-almost affine mapping of M onto M and $F : M \to \mathcal{K}(X)$ a continuous mapping such that:*

$$\inf_{x \in M} \alpha[\{y;\ y \in M,\ d(g(y), F(y)) \leq d(g(x), F(y))\}] = 0.$$

Then there exists $y_0 \in M$ *such that*

$$d(g(y_0), F(y_0)) = \inf_{x \in M} d(x, F(y_0)).$$

Proof. We shall prove that all the conditions of Theorem A are satisfied for $R(x)$ $(x \in M)$ defined by:

$$R(x) = \{y;\ y \in M,\ d(g(y), F(y)) \leq d(g(x), F(y))\}.$$

First, we shall prove that R is a $KKM_h(M)$ mapping. Let $\{x_1, x_2, \ldots, x_m\} \subset M$ and $y \in co_h\{x_1, x_2, \ldots, x_m\}$. If $y \notin \bigcup_{i=1}^{m} R(x_i)$ i.e. $y \notin R(x_i)$, $i \in \{1, 2, \ldots, m\}$ then

$$d(g(y), F(y)) > d(g(x_i), F(y)),\ i \in \{1, 2, \ldots, m\}. \tag{2}$$

From (2) it follows that there exists $\{v_1, v_2, \ldots, v_m\} \subset F(y)$ such that

$$d(g(y), F(y)) > d(g(x_i), v_i),\ i \in \{1, 2, \ldots, m\}. \tag{3}$$

Since $F(y)$ is compact and convex subset of X and g is generalized h-almost affine it follows that

$$d(g(y), F(y)) = \min_{z \in F(y)} d(g(y), z) \leq \max_{1 \leq i \leq m} d(g(x_i), v_i),$$

which contradicts (3). Hence

$$co_h\{x_1, x_2, \ldots, x_m\} \subset \bigcup_{i=1}^{m} R(x_i)$$

which means that $R \in KKM_h(M)$. In order to prove that $R(x)$ is closed for every $x \in M$ we shall prove that the mapping $y \to d(g(y), F(y))$ $(y \in M)$ is lower semicontinuous and for every $x \in M$, $y \to d(g(x), F(y))$ is upper semicontinuous.

Since $F(y)$ is compact it follows that for $\gamma > 0$:

$$\begin{aligned} P_\gamma = \ & \{y;\ y \in M,\ d(g(y), F(y)) > \gamma\} = \\ & \{y;\ y \in M,\ (g(y), F(y)) \subset \{(z, v);\ (z, v) \in M \times X;\ d(z, v) > \gamma\}\}. \end{aligned}$$

The set $\{(z,v);\ (z,v) \in M \times X;\ d(z,v) > \gamma\}$ is open and the mapping $y \to (g(y), F(y))$ is upper semicontinuous, hence P_γ is open and so $y \to d(g(y), F(y))$ is lower semicontinuous.

Similarly, if

$$\begin{aligned} Q_\gamma = \quad & \{y;\ y \in M,\ d(g(x), F(y)) < \gamma\} = \\ & \{y;\ y \in M,\ F(y)) \cap \{v;\ v \in X;\ d(g(x), v) < \gamma\} \neq \emptyset\} \end{aligned}$$

then Q_γ is open, since F if lower semicontinuous and $\{v;\ v \in X;\ d(g(x), v) < \gamma\}$ is open.

Hence $y \to d(g(x), F(y))$ $(y \in M)$ is upper semicontinuous.

From Theorem A it follows that $\bigcap_{x \in M} R(x) \neq \emptyset$. If $y_0 \in R(x)$, for every $x \in M$ then

$$d(g(y_0), F(y_0)) \leq d(g(x), F(y_0))$$

for every $x \in M$ and so

$$d(g(y_0), F(y_0)) = \inf_{x \in M} d(x, F(y_0)).$$

Corollary 1. *Let $(X, \|\cdot\|)$ be a normed space, $\emptyset \neq M$ a convex and complete subset of X, g a continuous mapping from M onto M such that (b) holds and $F : M \to \mathcal{K}(X)$ a continuous mapping such that*

$$\inf_{x \in M} \alpha[\{y;\ y \in M,\ d(g(y), F(y)) \leq d(g(x), F(y))\}] = 0.$$

Then there exists $y_0 \in M$ such that

$$d(g(y_0), F(y_0)) = \inf_{x \in M} d(x, F(y_0)).$$

Corollary 2. *Let (X, d, W) be a convex metric space such that (d) holds, and W is continuous or $co_W(A)$ is compact for every finite $A \subset X$. Let $\emptyset \neq M$ be a convex and complete subset of X, and $F : M \to \mathcal{K}(X)$ a continuous mapping such that*

$$\inf_{x \in M} \alpha[\{y;\ y \in M,\ d(y, F(y)) \leq d(g(x), F(y))\}] = 0.$$

Then there exists $y_0 \in M$ such that

$$d(y_0, F(y_0)) = \inf_{x \in M} d(x, F(y_0)). \tag{4}$$

Proof. Since W is continuous or $co_W(A)$ is compact for every finite $A \subset X$ it follows that $W|co_W(A) \times co_W(A) \times [0,1]$ is continuous and so (X, d, W) is a pseudoconvex metric space, where $h = W$. From Lemma 2 it follows that $g(x) = x$, $x \in M$ is a generalized h- almost affine mapping and so from Theorem 1 it follows the existence of an element $y_0 \in M$ such that (4) holds.

Remark. If in Theorem 1 we suppose that $F : M \to \mathcal{K}(X)$ so that $F(y) \cap M \neq \emptyset$, for every $y \in M$ then we obtain the existence of an element $y_0 \in M$ such that $g(y_0) \in F(y_0)$.
Hence, if in Corollary 2 we suppose that $F(y) \cap M \neq \emptyset$, for every $y \in M$ we obtain that $y_0 \in F(y_0)$.

4. A theorem on approximate fixed points

In the next theorem $N_\varepsilon(K) = \{x;\ x \in X,\ d(x, K) < \varepsilon\}$ $(K \subset X, \varepsilon > 0)$.

Theorem 2. *Let (X, d, W) be a convex metric space such that (d) holds, and $co_W(A)$ is compact for every finite $A \subset X$. Let $\emptyset \neq M$ be a convex and complete subset of X, $\varepsilon > 0$ and $F : M \to 2^{N_\varepsilon(M)} \cap \mathcal{K}(X)$ a continuous mapping such that $F(M)$ is bounded. Then*

$$\inf_{x \in M} d(x, F(x)) \leq \varepsilon + \alpha[F(M)]. \tag{5}$$

Proof. Let $\delta > 0$ and $\{u_1, u_2, \ldots u_n\} \subset F(M)$ $(i \in \{1, 2, \ldots, n\})$ be an $\alpha[F(M)] + \frac{\delta}{2}$-net of the set $F(M)$. Let $u_i \in F(x_i)$, $i \in \{1, 2, \ldots, n\}$. Since $\{u_1, u_2, \ldots u_n\}$ is an $\alpha[F(M)] + \frac{\delta}{2}$-net of the set $F(M)$

$$F(M) \subseteq \bigcup_{i=1}^{n} L(u_i,\ \alpha[F(M)] + \frac{\delta}{2})$$

and from $F(x_i) \subseteq N_\varepsilon(M)$,$i \in \{1, 2, \ldots, n\}$ it follows the existence of $\{v_1, v_2, \ldots v_n\} \subset M$ such that

$$d(u_i, v_i) < \varepsilon + \frac{\delta}{2},\ i \in \{1, 2, \ldots, n\}.$$

The set $H = co_W(\{x_1, x_2, \ldots x_n, v_1, v_2, \ldots v_n\})$ is a compact and convex subset of M and from Corollary 2 it follows the existence of an $y_0 \in H$ such that

$$d(y_0, F(y_0)) = \inf_{x \in H} d(x, F(y_0)). \tag{6}$$

We shall prove that $\inf_{x \in H} d(x, F(y_0)) \leq \varepsilon + \alpha[F(M)]$ which implies (5). For every $u \in F(y_0)$ there exists $u_i (i \in \{1, 2, \ldots, n\})$ such that $u_i \in F(x_i)$ and $d(u, u_i) < \alpha[F(M)] + \frac{\delta}{2}$, and v_i $(i \in \{1, 2, \ldots, n\})$ such that $d(u_i, v_i) < \varepsilon + \frac{\delta}{2}$. Then $d(u, v_i) < \alpha[F(M)] + \varepsilon + \delta$ and so $d(v_i, F(y_0)) < \alpha[F(M)] + \varepsilon + \delta$. Hence

$$\inf_{x \in H} d(x, F(y_0)) < \alpha[F(M)] + \varepsilon + \delta.$$

and since δ is an arbitrary positive number we obtain (5).

Corollary 3. *Let (X, d, K) be a strongly convex metric space and for $W_K(d)$ holds. Let $\emptyset \neq M$ be a convex and complete subset of X, $\varepsilon > 0$ and $F : M \to 2^{N_\varepsilon(M)} \cap \mathcal{K}(X)$ a continuous mapping such that $F(M)$ is bounded. Then (5) holds.*

Proof: In a strongly convex metric space $co_{W_K}(A)$ is compact, for every finite subset of X.

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REZIME

O NAJBOLJIM APROKSIMACIJAMA ZA VIŠEZNAČNA PRESLIKAVANJA U PSEUDOKONVEKSNIM METRIČKIM PROSTORIMA

Dokazano je uopštenje Ky Fanove [1] teoreme o najboljim aproksimacijama za višeznačna preslikavanja u pseudokonveksnim metričkim prostorima.

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