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ON A HADŽIĆ'S COMMON FIXED POINT THEOREM IN 2-METRIC SPACES

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Abstract

In this note we give a result similar to Hadžić's common fixed point theorem [3, Th. 1] on metric spaces for the case of 2-metric spaces which is a generalization of Theorem 1 from [2]. Also an extension of Sehgal's theorem [8] is obtained.

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In [1] S.Gähler introduced the notion of 2-metric space: Let X be an arbitrary set and d a real valued function on $X \times X \times X$ satisfying the following conditions:

i) To each pair of points $(x, y) \in X \times X$ with $x \neq y$ there is $z \in X$ such that $d(x, y, z) \neq 0$.

ii) $d(x, y, z) = 0$ only when at least two of three points are equal.

iii) For every $x, y, z \in X$: $d(x, y, z) = d(x, z, y) = d(y, z, x)$.

iv) For every $x, y, z, u \in X$:

$$d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z).$$

Then (X, d) is said to be a 2-metric space. The convergence in a 2-metric space (X, d) is introduced in the following way [2]. Let $\{x_n\}_{n \in \mathbf{N}}$ be a sequence from X and $x \in X$. We say that the sequence $\{x_n\}_{n\in\mathbf{N}}$converges to x if and only if for every $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$. The sequence $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence if $\lim_{m,n\to\infty} d(x_n, x_m, a) = 0$ for all $a \in X$. If every Cauchy sequence in X converges to a point in X we say that (X, d) is a complete 2-metric space.

Some fixed point theorem in such spaces are proved in [2], [4], [5] and [6].

In the proof of our theorem we shall use the following Lemma proved in [7].

Lemma. *Let* $Q_q = \{g | g \in \mathbf{R}^+,\ g \leq q + f(g)\}$ *for every* $q \in \mathbf{R}^+$, *where* $f : [0, \infty) \to [0, \infty)$ *is a given non - decreasing function such that* $\lim_{n\to\infty} f^n(g) = 0$ *for* $g > 0$ *and* $\lim_{g\to\infty}(g - f(g)) = \infty$. *Then*

1) $Q_q \neq \emptyset$ *and* $\hat{f}(Q_q) \subseteq Q_q$ *where* $\hat{f}(g) = q + f(g),\ g \geq 0$.

2) Q_q *is bounded for each* $q > 0$ *and the maximal solution* $m(q) = \sup_{t\in Q} t$ *of the inequality* $g \leq q + f(g)$ *is a fixed point of* $\hat{f}$.

3) The maximal solution $m(0)$ *of the inequality* $g \leq f(g)$ *is equal to* 0.

Theorem 1. *Let* (X, d) *be a complete 2-metric space,* S *and* T *be one to one continuous mappings from* X *into* X, A *be a continuous mapping from* X *into* $SX \cap TX$ *and* A *commute with* S *and* T. *Suppose that the following conditions are satisfied;*

1) For every $x \in X$ *there exists* $n(x) \in \mathbf{N}$ *so that for every* $y \in X$ *and* $a \in X$:

$$d(A^{n(x)}x, A^{n(x)}y, a) \leq \min\{q(d(Sx, Ty, a))d(Sx, Ty, a), d(Tx, Sy, a)\}$$

where $q : [0, \infty) \to [0, 1)$ *is a non-decreasing function such that*

$$\lim_{t\to\infty} t(1 - q(t)) = \infty$$

2) For some $x_0 \in X$ *we have that the sets*

$$M = \{A^m T^p x_0 \mid p \in \mathbf{N},\ m \in \{0, 1, ..., n(x_0) - 1\}\}$$

$$U = \{A^m T^p (TS)^{-s} x_0 \mid s \in \mathbf{N},\ p \in \mathbf{N} \cup \{0, -1\},\ m \in \{0, 1, ..., n(x_0) - 1\}\}$$

are so that for every $a, b \in X$ *the sets*

$$\{d(x, a, b) \mid x \in M\},\ \{d(y, a, b) \mid y \in U\},\ \{d(x, a, y) \mid x \in M,\ y \in U\}$$

are bounded.

Then there exists one and only one element $y \in X$ such that

$$y = Sy = Ty = Ay.$$

Proof. We shall prove that the set

$$B = \{d(A^n T^k x_0, Sx_0, a) \mid n, k \in \mathbf{N} \cup \{0\}\}$$

is bounded.

Let $n = p \cdot n(x_0) + r$ where $0 \le r < n(x_0)$. Let us prove that

$$(*) \qquad d(A^{p \cdot n(x_0)+r} T^k x_0, Sx_0, a) \le b_p(a)$$

for every $p \in \mathbf{N}$ and $k \in \mathbf{N} \cup \{0\}$ where

$$b_0(a) = \sup_{t \in M \cup \{A^{n(x_0)} x_0\}} d(t, Sx_0, a)$$

and

$$b_p(a) = b_0(a) + q(b_{p-1}(a)) b_{p-1}(a), \quad p \in \mathbf{N}.$$

We will prove this by induction in respect to $p \in \mathbf{N}$.

For $p = 1$ we have that

$$d(A^{n(x_0)+r} T^k x_0, Sx_0, a) \le d(A^{n(x_0)} x_0, Sx_0, a)$$

$$+ d(A^{n(x_0)+r} T^k x_0, A^{n(x_0)} x_0, a) + d(A^{n(x_0)+r} T^k x_0, Sx_0, A^{n(x_0)} x_0$$

$$\le b_0(a) + q(d(Sx_0, A^r T^{k+1} x_0, a)) d(Sx_0, A^r T^{k+1} x_0, a)$$

$$+ q(d(Sx_0, A^r T^{k+1} x_0, Sx_0)) d(Sx_0, A^r T^{k+1} x_0, Sx_0)$$

$$\le b_0(a) + q(b_0(a)) b_0(a) = b_1(a).$$

Suppose that $(*)$ is satisfied for some $p \in \mathbf{N}$ and every $k \in \mathbf{N} \cup \{0\}$ and prove that

$$d(A^{(p+1)n(x_0)+r} T^k x_0, Sx_0, a) \le b_{p+1}(a)$$

for every $k \in \mathbf{N} \cup \{0\}$.

We have that

$$d(A^{(p+1)n(x_0)+r}T^k x_0, Sx_0, a) \leq d(A^{n(x_0)}x_0, Sx_0, a)$$

$$+d(A^{(p+1)n(x_0)+r}T^k x_0, A^{n(x_0)}x_0, a) + d(A^{(p+1)n(x_0)+r}T^k x_0, Sx_0, A^{n(x_0)}x_0)$$

$$\leq b_0(a) + q(d(Sx_0, A^{pn(x_0)+r}T^{k+1}x_0, a))d(Sx_0, A^{pn(x_0)+r}T^{k+1}x_0, a)$$

$$+q(d(Sx_0, A^{pn(x_0)+r}T^{k+1}x_0, Sx_0))d(Sx_0, A^{pn(x_0)+r}T^{k+1}x_0, Sx_0)$$

$$\leq b_0(a) + q(b_p(a))b_p(a) = b_{p+1}(a).$$

Considering $f(t) = tq(t)$, $t \geq 0$, we have $f^n(t) \leq t(q(t))^n$ for every $t \geq 0$ and since $q(t) < 1$ for every $t \geq 0$ we deduce that $\lim_{n\to\infty} f^n(t) = 0$. We have $\lim_{t\to\infty}[t - f(t)] = \lim_{t\to\infty} t(1 - q(t)) = \infty$ and so we may apply the Lemma. Let $\hat{f}(t) = b_0(a) + f(t)$ for $t \geq 0$. Because $b_0(a) \leq \hat{f}(b_0(a))$ we deduce that $b_0(a) \in Q_{b_0(a)}$ and since $\hat{f}(Q_{b_0(a)}) \leq Q_{b_0(a)}$ it follows that $\{b_p(a) \mid p \in \mathbf{N}\} \subseteq Q_{b_0(a)}$ and so the sequence $\{b_p(a)\}_{p\in\mathbf{N}}$ is bounded.

So we have proved that the set B is bounded and let

$$D(a) = \sup_{n,k\in\mathbf{N}\cup\{0\}} d(A^n T^k x_0, Sx_0, a).$$

Similarly, we obtain that the set

$$\{d(A^n T^k (TS)^{-s} x_0, Sx_0, a) \mid n, s \in \mathbf{N} \cup \{0\},\ k \in \mathbf{N} \cup \{0, -1\}\}$$

is bounded and let

$$E(a) = \sup_{n,s\in\mathbf{N}\cup\{0\} k\in\mathbf{N}\cup\{0,-1\}} d(A^n T^k (TS)^{-s} x_0, Sx_0, a).$$

We shall prove that the set

$$V = \{d(x, Sx_0, y) \mid x \in \{A^n T^k x_0 \mid n, k \in \mathbf{N} \cup \{0\}\},$$

$$y \in \{A^n T^k (TS)^{-s} x_0,\ \ n, s \in \mathbf{N} \cup \{0\},\ \ k \in \mathbf{N} \cup \{0, -1\}\}\}$$

is bounded.

Let $n = pn(x_0) + r$, $n' = p'n(x_0) + r'$ where $0 \leq r$, $r' < n(x_0)$,

$$x = A^{pn(x_0)+r}T^k x_0,\ \ y = A^{p'n(x_0)+r'}T^{k'}(TS)^{-s}x_0.$$

We have that

$$d(A^{pn(x_0)+r}T^k x_0, Sx_0, A^{p'n(x_0)+r'}T^{k'}(TS)^{-s}x_0)$$
$$\leq d(A^{n(x_0)}x_0, Sx_0, A^{p'n(x_0)+r'}T^{k'}(TS)^{-s}x_0)$$
$$+q(d(Sx_0, A^{(p-1)n(x_0)+r}T^{k+1}x_0, A^{p'n(x_0)+r'}(TS)^{-s}x_0))$$
$$\cdot d(Sx_0, A^{(p-1)n(x_0)+r'}T^{k+1}x_0, A^{p'n(x_0)+r'}T^{k'}(TS)^{-s}x_0)$$
$$\leq q(d(Sx_0, A^{(p'-1)n(x_0)+r'}T^{k'+1}(TS)^{-s}x_0, Sx_0))$$
$$\cdot d(Sx_0, A^{(p'-1)n(x_0)+r'}T^{k'+1}(TS)^{-s}x_0, Sx_0)$$
$$+d(Sx_0, A^{(p-1)n(x_0)+r}T^{k+1}x_0, A^{p'n(x_0)+r'}T^{k'}(TS)^{-s}x_0)$$
$$= d(Sx_0, A^{(p-1)n(x_0)+r}T^{k+1}x_0, A^{p'n(x_0)+r'}T^{k'}(TS)^{-s}x_0).$$

We deduce so that

$$d(x, Sx_0, y) \leq d(A^r T^{k+p}x_0, Sx_0, A^{r'}T^{k'+p'}(TS)^{-s}x_0)$$

and so V is bounded because $\{d(x,a,y) \mid x \in M,\ y \in U\}$ is bounded.

As in [2] let $x_0 \in X$. Since $AX \subseteq SX \cap TX$ we can define the sequence $\{x_n\}_{n\in\mathbf{N}}$ such that

$$Tx_{2k-1} = A^{n(x_{2k-2})}x_{2k-2},\ \ Sx_{2k} = A^{n(x_{2k-1})}x_{2k-1},\quad k \in \mathbf{N}.$$

We shall prove that $\{y_m\}_{m\in\mathbf{N}}$ is a Chauchy sequence where

$$y_m = \begin{cases} Tx_{2k-1}, & m = 2k-1, \\ Sx_{2k}, & m = 2k,\quad k \in \mathbf{N}. \end{cases}$$

By a similar reason as in [2] and [3] we obtain that for every $m, k \in \mathbf{N},\ k > 2$ and $a \in X$

$$d(A^{n(x_{2k-3})}x_{2k-3}, A^m x_{2k-2}, a) \leq (q(d(Sx_0, A^m x_0, a))^{k-1} \cdot d(Sx_0, A^m x_0, a) \tag{1}$$

and so

$$d(y_{2k-1}, y_{2k}, a) \leq$$
$$\leq (q(d(Sx_0, A^{n(x_{2k-1})}x_0, a)))^k d(Sx_0, A^{n(x_{2k-1})}x_0, a)$$
$$\leq q^k(D(a))D(a).$$

Also we have

$$d(y_{2k},y_{2k+1},a)=d(Sx_{2k},Tx_{2k+1},a)\le q^k(D(a))\cdot D(a).$$

We deduce that

$$d(y_n,y_{n+1},a)\le (q(D(a)))^{[\frac{n+1}{2}]}\cdot D(a),\quad n\in \mathbf{N}.$$

Since

$$d(y_n,y_{n+m},a)\le d(y_n,y_{n+1},y_{n+m})$$
$$+d(y_n,y_{n+1},a)+d(y_{n+1},y_{n+2},y_{n+m})$$
$$+d(y_{n+1},y_{n+2},a)+\dots+d(y_{n+m-2},y_{n+m-1},y_{n+m})$$
$$+d(y_{n+m-1},y_{n+m},a)\le \sum_{k=n}^{n+m-2}(q(D(y_{n+m})))^{[\frac{k+1}{2}]}\cdot D(y_{n+m})$$
$$+\sum_{k=n}^{n+m-1}(q(D(a)))^{[\frac{k+1}{2}]}\cdot D(a)\le \sum_{k=n}^{n+m-2}(q(D))^{[\frac{k+1}{2}]}\cdot D$$
$$+\sum_{k=n}^{n+m-1}(q(D(a)))^{[\frac{k+1}{2}]}\cdot D(a)$$

where D is an upper bound for $D(y_{n+m})$, $n,m\in\mathbf{N}$.

It follows that $\{y_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence. So there exists $y\in X$ such that $\lim_{n\to\infty}y_n=y$ which means that for every $a\in X$

$$\lim_{n\to\infty}d(y_n,y,a)=0.$$

Since $\{Sx_{2k}\}_{k\in\mathbf{N}}$ and $\{Tx_{2k-1}\}_{k\in\mathbf{N}}$ are subsequences of the sequence $\{y_n\}_{n\in\mathbf{N}}$ it follows that for every $a\in X$

(2) $$\lim_{k\to\infty}d(Sx_{2k},y,a)=\lim_{k\to\infty}d(Tx_{2k-1},y,a)=0.$$

We have that

$$d(Sx_{2k},Ax_{2k},a)=d(A^{n(x_{2k-1})}x_{2k-1},Ax_{2k},a)$$
$$\le (q(d(Sx_0,Ax_0,a)))^k d(Sx_0,Ax_0,a)\le (q(D(a)))^k D(a)$$

for every $k\in\mathbf{N}$ and so

$$d(Ax_{2k},y,a)\le d(Ax_{2k},y,Sx_{2k})+d(Ax_{2k},Sx_{2k},a)+d(Sx_{2k},y,a)$$
$$\le (q(D(a)))^kD(a)+(q(D(y)))^kD(y)+d(Sx_{2k},y,a).$$

Using (2) we conclude that

$$\lim_{k\to\infty} d(Ax_{2k},y,a)=0$$

for every $a\in X$.

Furthermore,

$$d(Sx_{2k},A^2x_{2k},a)=d(A^{n(x_{2k-1})}x_{2k-1},A^2x_{2k},a)$$
$$\le (q(d(Sx_0,A^2x_0,a)))^kd(Sx_0,A^2x_0,a)$$
$$\le (q(D(a)))^kD(a) \quad \text{for every} \quad k\in \mathbf{N}$$

which implies that

$$\lim_{k\to\infty} A^2x_{2k}=y.$$

Then from

$$d(Ay,Sy,a)\le d(Ay,Sy,ASx_{2k})+d(Ay,ASx_{2k},a)+d(ASx_{2k},Sy,a)$$
$$=d(Ay,ASx_{2k},Sy)+d(Ay,ASx_{2k},a)+d(SAx_{2k},Sy,a)$$

since A and S are continuous, we deduce that

$$d(Ay,Sy,a)=0,\quad (\forall)a\in X.$$

Hence $Ay=Sy$.

Similarly we can prove that $Ay=Ty$.

Since $y=\lim_{k\to\infty}A^2x_{2k}=A(\lim_{k\to\infty}Ax_{2k})=Ay$ we obtain that y is a common fixed point for the mappings A,S and T. It is easy to see that y is the unique common fixed point.

Remark. Let us consider the case $Tx=Sx+x$ for all $x\in X$. Then every continuous function A which satisfies condition 1) of the theorem satisfies also 2) (these sets are finite) so that we can apply the theorem to deduce that A has a unique fixed point.

This is the corresponding extension of the theorem of Sehgal [8] in 2 - metric spaces.

The remark from ([2], p. 11) asserts that instead of the boundedness of the space X we can suppose that for every $a \in X$ we have

$$\sup_{\substack{y \in O_A(a) \\ z \in X}} d(a,y,z) \leq M_a, \quad O_A = \{A^n a, \ n \in \mathbf{N} \cup \{0\}\}$$

In the above theorem this condition does not appear.

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REZIME

O TEOREMI O. HADŽIĆ O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI U 2-METRIČKIM PROSTORIMA

U radu se daje rezultat sličan teoremi O. Hadžić o zajedničkoj nepokretnoj tački u metričkim prostorima za slučaj 2-metričkih prostora koji je generalizacija Teoreme 1 iz [2]. Takodje se daje proširenje Sehgal-ove teoreme [8].

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