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NONOSCILLATORY SOLUTIONS OF THE SECOND ORDER NONLINEAR DELAY DIFFERENCE EQUATION

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Abstract

The paper deal with the second order nonlinear delay difference equation and gives conditions which insure the nonexistence of one of three possible type of nonoscillatory solutions.

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1. Introduction

We are concerned with the second order nonlinear delay difference equation of the form

$$(*) \qquad \Delta^2 x(n) + p(n) f(x(\tau(n))) = 0, \quad n = 1, 2, \ldots$$

and supose that

(1) $p(n) \geq 0$ and $p(n)$ is not eventually zero,

(2) $\tau(n) \in N, \quad \tau(n) \leq n$ and $\lim_{n \to \infty} = \infty$,

(3) $sf(s) > 0$ for $s \neq 0$.

Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$.

By a solution of $(*)$ we mean a real sequence $\{x(n)\}$ satisfying $(*)$. Throughout this paper, we usually refer to a solution $\{x(n)\}$ simply as a solution x and consider only nontrivial solutions.

A real sequence $\{x(n)\}$ has some property eventually if there exist $N \geq 1$ such that $x(n)$ has this propert for $n = N, N+1, \dots$.

A nontrivial solution x of $(*)$ is said to be oscillatory if $x(n)$ changes sign infinitely many times. Otherwise, x is said to be nonoscillatory.

There is a current interest in oscillation and nonoscillation theory of differential equations, but not to much is done in the theory of difference equations. This, and the fact that the results for corresponding difference equation could be quite different, are motivations for this paper.

2. Preliminaries

In what follows we shall use the following lemma which gives useful information about the bounds for nonoscillatory solutions of the $(*)$ equation.

Lemma 1. *[1] Consider $(*)$ subject to the conditions (1), (2) and (3). Then, every nonoscillatory solution x of $(*)$ satisfies eventually the following estimate*

$$A \leq |x(n)| \leq Bn$$

for some positive constants A and B (depending on x).

Beside this "a priori" estimate we need the next

Lemma 2. *Let $g(t)$ be a non-negative function defined on some neighbourhood of infinity and let k be a positive real number. If*

$$\lim_{n\to\infty} \sup n^k \sum_{s=n}^{\infty} g(s) = \infty$$

then

$$\sum_{n=n_0}^{\infty} n^k g(n) = \infty.$$

Proof. It is an immediate consequence of the following inequalities

$$n^k \sum_{s=n}^{\infty} g(s) \leq \sum_{s=n}^{\infty} s^k g(s) \leq \sum_{n=n_0}^{\infty} n^k g(n).$$

3. Nonoscillatory solutions

Lemma 1 implies that for nonoscillatory solutions of $(*)$ there are only three possibilities:

a) $|x(n)| \to C, \quad \Delta x(n) \to 0, \ C > 0,$

b) $|x(n)| \to \infty, \quad \Delta x(n) \to 0,$

c) $|x(n)| \to \infty, \quad |\Delta x(n)| \to 0, \ C > 0.$

Noe we are able to give the following theorem.

Theorem 1. *Consider $(*)$ subject to the conditions (1), (2), (3),*

(4) f is nondecreasing and

$$\sum_{n=n_0}^{\infty} n f(\tau(n)) p(n) < \infty.$$

Then, $(*)$ has not a nonoscillatory solution of type b).

Proof. Let x be a nonoscillatory solution of type b). Without loss of generality we may suppose that $x(n) > 0$ eventually. Then $x(\tau(n)) > 0$, $\Delta x(n) \geq 0$ and $\Delta^2 x(n) \leq 0$ eventually.

Summing equation $(*)$ from n to ∞ we get

$$\Delta x(n) = \sum_{s=n}^{\infty} p(s) f(x(\tau(s))).$$

Since the sequence x is increasing and $\lim_{n\to\infty} \Delta x(n) = 0$, for every $A > 0$ we have $x(n) \leq An$ eventually and also $f(x(\tau(n))) \leq f(\tau(n))$ eventually. Therefore

$$\Delta x(n) \leq \sum_{s=n}^{\infty} p(s) f(\tau(s))$$

eventually. Summing the above inequalities from n_0 to n we get

$$x(n+1) \le x(n_0) + \sum_{k=n_0}^{n} \sum_{s=k}^{\infty} p(s) f(\tau(s)) =$$

$$= x(n_0) + \sum_{s=n_0}^{n} (s+1-n_0) p(s) f(\tau(s)) + (n+1-n_0) \sum_{s=n+1}^{\infty} p(s) f(\tau(s)).$$

When n tends to infinity the left side of last inequality tends to infinity while the right side is, according to Lemma 2, finite. This contradiction completes the proof.

Under some sharper condition on the function f we get the next result.

Theorem 2. *Consider* $(*)$ *subject to the conditions (1), (2), (3),*

(5) $\frac{f(x)}{x^\alpha}$ *is nondecreasing for some* $\alpha > 0$ *and*

(6) $S(n) = np(n)(\tau(n))^\alpha$ *is nondecreasing.*

Then, $(*)$ *has not a nonoscillatory solution of type c).*

Proof. Let x be a nonoscillatory solution of $(*)$. Without loss of generality we may suppose that $x(n) > 0$ eventually. Then $x(\tau(n)) > 0$ eventually which implies that $\Delta^2 x(n) \le 0$, $\Delta x(n) \ge 0$ eventually and $\lim_{n\to\infty} \Delta x(n) \ge 0$.

Suppose that $\lim_{n\to\infty} \Delta x(n) = b > 0$ and as a consequence $\Delta x(n) \ge \frac{b}{2}$ eventually. It follows that $x(n) \ge \frac{bn}{2}$ and $x(\tau(n)) \ge \frac{b}{2}\tau(n)$ eventually. Then

$$-\Delta^2 x(n) = p(n) f(x(\tau(n))) \ge K_1 p(n) \tau^\alpha(n).$$

Let us choose n_1 great enough such that $S(n_1) > 0$. After summation the above inequalities from n_1 to n, we get

$$\Delta x(n_1) - \Delta x(n) \ge K_1 \sum_{s=n_1}^{n} p(s) \tau^\alpha(s) \ge K_1 S(n_1) \sum_{s=n_1}^{n} \frac{1}{s},$$

which implies that

$$\Delta x(n) \le \Delta x(n_1) - K_2 \sum_{s=n_1}^{n} \frac{1}{s}.$$

As the right side of the last inequality tends to $-\infty$, while $n \to \infty$, we get $\Delta x(n) < 0$ eventually and this contradiction completes the proof.

Without any difficulties we can prove the theorems for the adequate differential equation. In this case Theorem 1 is an improvement of Theorem 1 [2]. Theorem 2 is also an improvement of Theorem 5 [3].

References

[1] Kulenović, M.R.S., Budinčević, M., Asymptotic analysis of nonlinear second order difference equation, An. Stiin. Univ. "Al. I. Cuza", tom XXX (1984), 39-52.

[2] Ohriska, J., Nonoscillatory solutions of a second order nonlinear delay differential equation, Math. Slovaca 35, (1985), 105-110.

[3] Ohriska, J., Asymptotic properties of solutions of a second order nonlinear delay differential equation, Math. Slovaca 31, (1981), 83-90.

REZIME

NEOSCILATORNA REŠENJA NELINEARNE DIFERENCNE JEDNAČINE DRUGOG REDA SA KAŠNJENJEM

U radu se posmatra nelinearna diferencna jednačina drugog reda sa kašnjenjem, oblika

$$\Delta^2 x(n) + p(n)f(x(\tau(n))) = 0, \quad n = 1, 2, \ldots$$

uz uslove:

(1) $p(n) \geq 0$ i $p(n)$ nije eventualno jednako nuli,

(2) $\tau(n) \in N, \quad \tau(n) \leq n, \quad \lim_{n\to\infty} = \infty$,

(3) $sf(s) > 0$ za $s \neq 0$.

Daju se dovoljni uslovi da njena neoscilatorna rešenja ne bi pripadala jednoj od tri moguće klase ponašanja.

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