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AN APPLICATION OF THE GENERALIZED B–TRANSFORM

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Abstract

The equation $xu_{xx}(x,y)+2u_x(x,y)+Au_{yy}(x,y) = f(x,y)$, $f(x,y) = \sum_{n=0}^{\infty} a_n(y)l_n \in E(\mathbf{R}_+, LG'_0)$ (resp. $E(\mathbf{R}_+, LG'_e)$), $A \in \mathbf{R}$ is considered. We solve the corresponding boundary value problem in $E(\mathbf{R}_+, LG'_e)$ (Prop. 1). If f has the expansion in x of appropriate form and A is a smooth function or a constant than we find the Laguerre series solution (Prop. 2). If A is a smooth function on $(0,\infty)$ we solve this equation in $E(\mathbf{R}_+, LG'_0)$, (resp. $E(\mathbf{R}_+, LG'_e)$), assuming that $a_n, n \in \mathbf{N_0}$, and the first coefficient of the Laguerre series solution is the polynomial of arbitrary but fixed degree (resp. of first degree).

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1. Introduction

In [9] is given an operational formula for the B–transform or the Bessel transform which is analogous to the corresponding formula for the Hankel transform (see [1], [7]). By using it and the Laguerre series expansions we develope in this paper and [10] the corresponding operational calculus.

In Section 2. we define the $B-$transform and its Laguerre representation on the spaces LG'_0, LG'_e. In Section 3. we apply the $B-$transform in solving the equation

$$xu_{xx}(x,y)+2u_x(x,y)+Au_{yy}(x,y)=f(x,y),$$

$f(x,y)\in E(\mathbf{R}_+,LG'_0)$ (resp.$E(\mathbf{R}_+,LG'_e)$), with the appropriate boundary value conditions. Finally, in Section 4. we solve this equation for $A\in\mathbf{R}$ by using the properties of the $B-$transform and Laguerre polynomials. We suppose that

$$f(x,y)=\sum_{n=0}^{\infty}a_n(y)l_n(x),\;\; u(x,y)=\sum_{n=0}^{\infty}x_n(y)l_n(x),$$

where

$$x_n=\sum_{k=0}^{m}x_{n,k}x^k, n\in\mathbf{N_0}.$$

We consider separately the cases when $m=1, m=5$, and $m\in\mathbf{N_0}$. In the case $m=1$ we assume that A is a smooth function on $(0,\infty)$.

In the Appendix we give the b and $B-$transform for some elements from LG_0, LG_e and their duals with the help of Laguerre evaluation. We use the results from [11] , [1].

2. Generalized $B-$transform

Let $l_n=e^{-x/2}L_n, n\in\mathbf{N_0}$, where the Laguerre polynomials are

$$L_n(x)=\sum_{k=0}^{n}\binom{n}{n-k}(-x)^k/k!, x>0, n\in\mathbf{N_0}.$$

The operator $\mathcal{R}=e^{x/2}Dxe^{-x}De^{x/2}$ is self-adjoint, positive, $-n$, are its eigenvalues and $l_n, n\in\mathbf{N_0}$, are its eigenfunctions which constitutes the orthonormal base for $L^2(\mathbf{R}_+)$. The space LG_0 is defined as the space of functions $\varphi=\sum_{n=0}^{\infty}a_nl_n$ from $L^2(\mathbf{R}_+)$ for which $\sum_{n=0}^{\infty}|a_n|^2n^{2k}<\infty$ for every $k>0$, and $<\mathcal{R}\varphi, l_n>=<\varphi,\mathcal{R}l_n>, n\in\mathbf{N_0}$, where

$$<\varphi,\bar{\psi}>=(\varphi,\psi)=\int\varphi\bar{\psi}dt,\ \varphi,\ \psi\in L^2(\mathbf{R}_+).$$

In fact LG_0 is (topologically) equal to the space S_+ of smooth functions for which all the seminorms

$$||\phi||_{k,n} = \sup_{t\in[0,\infty)} t^k|\phi^{(n)}(t)|, k, n \in \mathbf{N_0},$$

are finite. ([11], [2], [8]).

Define ([10]) LG_e as the subspace of LG_0 such that $\phi = \sum_{n=0}^{\infty} a_n l_n \in LG_e$ if for every $k > 0$ $\sum_{n=0}^{\infty} |c_n|^2 k^{2n} < \infty$.

The notation of dual spaces is LG_0', LG_e' respectively. LG_0' is in fact, the space S_+' of tempered distributions supported by $[0,\infty)$. It may be identifyed with the space of formal series

$$LG_0' = \{\sum_{n=0}^{\infty} b_n l_n, \ \ iff \ \sum_{n=0}^{\infty} |b_n|^2 n^{-2k} < \infty, \quad \text{for some } k > 0\}.$$

Similarly,

$$LG_e' = \{\sum_{n=0}^{\infty} d_n l_n, \ \ iff \ \sum_{n=0}^{\infty} |d_n|^2 k^{-2n} < \infty, \quad \text{for some } k > 0\}.$$

The transform $b : LG_0 \to LG_0$ (resp. $LG_e \to LG_e$) is defined in [9], [12] by

$$b[\phi](t) = - < \phi'(\tau), J_0(\sqrt{t\tau}) >= \phi(0) + 1/2 < \phi(\tau), \sqrt{t/\tau} J_0(\sqrt{t\tau}) >,$$

where J_0 is the Bessel function of zero order. (see [9].)

Then , for $f \in LG_0'$ (resp. LG_e') we define $B[f]$ as an element of LG_0' (resp. LG_e') by the following prescription:

$$< B[f], \phi >=< f, b[\phi] >, \phi \in LG_0(\text{resp.} \phi \in LG_e).$$

If $\phi = \sum_{n=0}^{\infty} a_n l_n \in LG_0$ (resp. LG_e), then

$$b[\phi](t) = -2\sum_{n=0}^{\infty}(-1)^n(2\sum_{i=n+1}^{\infty} a_i + a_n)l_n(t), t > 0,$$

and

$$B[f] = \sum_{n=0}^{\infty}[2\sum_{m=0}^{n-1}(-1)^m b_m + (-1)^n b_n]l_n, ([9]).$$

The B−transform posseses the properties which provide the possibility of applications. Namely, we have the following operational formula

$$B[xu_{xx} + 2u_x](t) = -t/4B[u](t),\ x > 0, \tag{1}$$

where $u \in LG'_0$ (resp. LG'_e)([9]).

We shall use this formula in the Sections to follow.

Recall, ([12], [9]),

$$B[f * g] = B[f] * B[g],\ \ f, g \in LG'_0(\text{resp}.LG'_e),$$

$$B[f_\alpha] = 4^\alpha f_{-\alpha},\ \ \alpha \in \mathbf{R},$$

where

$$f_\alpha = \begin{cases} \mathbf{H}(t)t^{\alpha-1}/\Gamma(\alpha) \text{ if } \alpha > 0, \\ \\ D^N f_{\alpha+N}(t) \text{ if } \alpha \leq 0,\ \ \alpha + N > 0,\ \ N \in \mathbf{N}, \end{cases}$$

where D is the distributional derivative, and H is the Heaviside function.

The Fouirer-Laplace transform for $\varphi \in LG_0$ is defined by

$$(\mathcal{L}\varphi)(z) = \int_0^\infty \varphi(t)e^{izt}dt, Imz > 0, Rez \in \mathbf{R},$$

and for $f \in LG'_0$ by

$$(\mathcal{L}f)(z) = < f(t), e^{izt} >,\ Imz > 0,\ Rez \in \mathbf{R}.$$

Let $y \mapsto u(\cdot, y)$ be a C^∞- mapping from $(0, \infty)$ into LG'_0 (resp. LG'_e). We denote the space of such functions by $E(\mathbf{R}_+, LG'_0)$ (resp. $E(\mathbf{R}_+, LG'_e)$).

3. Boundary value problems

We deal with the equation

$$xu_{xx} + 2u_x + Au_{yy} = f(x, y), x, y > 0 \tag{2}$$

where $A \in \mathbf{R}$, and $f(x, y) \in E(\mathbf{R}_+, LG_0)$ (resp. $f(x, y) \in E(\mathbf{R}_+, LG'_e)$), along with the common boundary conditions

$$u(x, 0) = 0, u(x, l) = 0, \tag{3}$$

or general boundary conditions of the third kind

$$\left\{ \begin{array}{l} \bar{\beta}u(x,0) + \bar{\alpha}u'(x,0) = 0, |\bar{\alpha}| + |\bar{\beta}| \neq 0, \bar{\alpha}\bar{\beta} \leq 0, \\ \bar{\eta}u(x,l) + \bar{\gamma}u'(x,l) = 0, |\bar{\gamma}| + |\bar{\delta}| \neq 0, \bar{\gamma}\bar{\eta} \geq 0. \end{array} \right. \tag{4}$$

Putting for $(t, y > 0)$

$$\tilde{u}(t,y) = B[u(x,y)](t) \tag{5}$$

and applying (1) we obtain

$$A\tilde{u}_{yy}(t,y) - t/4\tilde{u}(t,y) = \tilde{f}(t,y) \tag{6}$$

with the boundary condition which corresponds to (3)

$$\tilde{u}(t,0) = 0, \tilde{u}(t,l) = 0, \tag{7}$$

and the boundary conditions

$$\left\{ \begin{array}{l} \alpha\tilde{u}(t,0) + \beta\tilde{u}'(t,0) = 0, \\ \gamma\tilde{u}(t,l) + \eta\tilde{u}'(t,l) = 0, \end{array} \right. \tag{8}$$

which correspond to (4).$(\alpha = \bar{\alpha}/4, \beta = \bar{\beta}, \gamma = \bar{\gamma}/4, \eta = \bar{\eta}.)$

Really, the equivalent form for (4) is

$$\left\{ \begin{array}{l} \bar{\alpha}(u(x,0) * \delta'(x)) + \bar{\beta}u(x,0) = 0, \\ \bar{\gamma}(u(x,l) * \delta'(x)) + \bar{\eta}u(x,l) = 0. \end{array} \right.$$

Making $B-$transform and using $B[f*g] = B[f]*B[g]$, and $B[(f_\alpha)] = 4^\alpha f_{-\alpha}$ from [9] , [12] we obtain

$$\left\{ \begin{array}{l} \bar{\alpha}(\tilde{u}(t,0) * 4^{-1}\mathbf{H}(t)) + \bar{\beta}\tilde{u}(t,0) = 0, \\ \bar{\gamma}(\tilde{u}(t,l) * 4^{-1}\mathbf{H}(t)) + \bar{\eta}\tilde{u}(t,l) = 0. \end{array} \right.$$

After differentiating we obtain

$$\left\{ \begin{array}{l} \bar{\alpha}/4\tilde{u}(t,0) + \bar{\beta}\tilde{u}'(t,0) = 0, \\ \bar{\gamma}/4\tilde{u}(t,l) + \bar{\eta}\tilde{u}'(t,l) = 0. \end{array} \right.$$

Assume that $A > 0$.

Solving (6) as an equation with the constant coefficients we obtain

$$\tilde{u}(t,y) = C_1 e^{y/2\sqrt{t/A}} + C_2 e^{-y/2\sqrt{t/A}} + Y_p(t,y) \tag{9}$$

where $Y_p(t,y)$ is a particular solution of (6) which depends on $\tilde{f}(t,y)$.

From the boundary conditions (7) we find constants C_1 and C_2, such that

$$\tilde{u}(t,y) = e^{y/2\sqrt{t/A}}/(2sh(l/2\sqrt{t/A}))[Y_p(t,0)e^{-l/2\sqrt{t/A}} - Y_p(t,l)]$$

$$(10)\ + e^{-y/2\sqrt{t/A}}/(2sh(l/2\sqrt{t/A}))[Y_p(t,l) - e^{l/2\sqrt{t/A}}Y_p(t,0)] + Y_p(t,y),$$

In the case (8) the solution (9) has the constants

$$C_1 = 1/D[e^{-l/2\sqrt{t/A}}(-\gamma - \eta/2\sqrt{t/A})(\alpha Y_p(t,0) + \beta Y_p'(t,0)) -$$
$$(\alpha - \beta/2\sqrt{t/A})(\gamma Y_p(t,l) + \eta Y_p'(t,l))],$$
$$C_2 = 1/D[(\alpha + \beta/2\sqrt{t/A})(\gamma Y_p(t,l) + \eta/2Y_p'(t,l)) - e^{l/2\sqrt{t/A}}$$
$$(\gamma + \eta/2\sqrt{t/A})(\beta Y_p'(t,0) + \alpha Y_p(t,0))],$$

where

$$D = -e^{-l/2\sqrt{t/A}}(\alpha + \beta/2\sqrt{t/A})(\gamma - \eta/2\sqrt{t/A}) + e^{l/2\sqrt{t/A}}$$
$$(\alpha - \beta/2\sqrt{t/A})(\gamma + \eta/2\sqrt{t/A}).$$

When $A < 0$ then (6) has the general solution

$$\tilde{u} = C\cos(y/2\sqrt{t/|A|}) + B\sin(y/2\sqrt{t/|A|}) + Y_p(t,y).$$

With conditions (7) it gives

$$\tilde{u}(t,y) = -Y_p(t,0)\cos(y/2\sqrt{t/|A|}) + [(-Y_p(t,l) +$$
$$Y_p(t,0)\cos(l/2\sqrt{t/|A|}))/\sin(l/2\sqrt{t/|A|})]\sin(y/2\sqrt{t/|A|}) + Y_p(t,y).$$

In the case of boundary condition (4) we obtain that constants in the general solution are the following

$$C = -1/D_1[(\alpha Y_p(t,0) + \beta Y_p'(t,0))(\gamma\sin(l/2\sqrt{t/|A|}) + \eta/2\sqrt{t/|A|}$$
$$\cos(l/2\sqrt{t/|A|})) - \beta/2\sqrt{t/|A|}(\gamma Y_p(t,l) + \eta Y_p'(t,l))],$$
$$B = -1/D_1[\alpha(\gamma Y_p(t,l) + \eta Y_p'(t,l)) - (\alpha Y_p(t,0) + \beta Y_p'(t,0))(\gamma\cos(l/2\sqrt{t/|A|})$$

$$-\eta\sqrt{t/|A|}\sin(l/2\sqrt{t/|A|}))],$$

where

$$D_1=(\alpha\gamma+\beta\eta t/(4|A|)\sin(l/2\sqrt{t/|A|})-1/2\sqrt{t/|A|}(\alpha\eta-\beta\gamma)\cos(l/2\sqrt{t/|A|}).$$

The boundary conditions (8) with $\alpha=1,\beta=0,\gamma=0,\eta=1$, in homogeneous case give $C=0$ and $\cos(l/2\sqrt{t/|A|})=0$ which give the oscilatory solution

$$\tilde{u}_n(y)=B_n\sin\frac{(2n+1)\pi y}{2l}.$$

Proposition 1.

1. *For $A>0$ the solution $\tilde{u}$ belongs to $E((0,\infty),LG'_e)$.*

2. *For $A<0$ the solution $\tilde{u}$ belongs to $E((0,\infty),LG'_0)$.*

Proof. 1. The assertion follows from the fact that for every $\phi\in LG_e, y\in[y_0-\delta,y_0+\delta]\subset(0,\infty)$, and $B>1/(2\sqrt{A})(y_0-\delta)$,

$$\int_0^\infty|e^{\sqrt{y/2t/A}}\phi(t)|dt\le\int_0^\infty|e^{\sqrt{y/2t/A}}e^{-tB}||e^{tB}\phi(t)|dt$$

$$\le sup_{t\in[0,\infty)}\{e^{tA}|\phi(t)|\}\int_0^\infty e^{\sqrt{t/A}/2y}e^{-tB}dt.$$

Because of that , the solution of equation (2) is the inverse for $\tilde{u}(t,y)$, and it belongs to $E((0,\infty),LG'_e)$.

2.Since $\tilde{u}$ is in $E((0,\infty),LG'_0)$ it follows that $u\in E((0,\infty),LG'_0)$.

4. Laguerre series solutions of $xu_{xx}(x,y)+2u_x(x,y)+A(y)u_{yy}(x,y)=f(x,y)$

We solved in [9] the equation $xu_{xx}+2u_x+A(y)u_{yy}=f(x)$ where $A(y)$ is a C^∞ function on $(0,\infty)$ in a form of Laguerre series by using the $B-$transform and assuming that the first coefficient in the Laguerre expansion of the solution is a linear function. In this section we generalize the results from [9] by solving the equation

$$xu_{xx}(x,y) + 2u_{xx}(x,y) + A(y)u_{yy}(x,y) = f(x,y), \tag{11}$$

$f(x,y) \in E(\mathbf{R}_+, LG'_0)$ (resp. $E(\mathbf{R}_+, LG'_e)$), in Lageerre series form with the help of $B-$transform by considering two different cases (A) and (B).

(A) : $A(y), y \in (0,\infty)$, is a C^∞ function ,

$$f(x,y) = \sum_{n=0}^{\infty}(p_{n,1}y + p_{n,0})l_n(x),$$

and the first coefficients in the Laguerre expansion of the solution is

$$a_0 = k_{0,1}y + k_{0,0}.$$

(B) : $A(y) = A \in \mathbf{R}, y \in (0,\infty)$, ,

$$f(x,y) = \sum_{n=0}^{\infty}(\sum_{i=0}^{5} p_{n,i}y^i)l_n(t), \quad a_0(y) = \sum_{i=0}^{5} k_{0,i}y^i.$$

Also, we shall find the solution which satisfies the condition

$$a_0(y) = \sum_{i=0}^{m} k_{0,i}y^i$$

where $k_{0,i}, i = 0(1)m, m \in \mathbf{N_0}$, are given constants.

In both cases we suppose that the solution of (11) is of the form

$$u(x,y) = \sum_{n=0}^{\infty} a_n(y)l_n(x), x,y > 0.$$

Proposition 2. *The solution u of (11) belongs to $E(\mathbf{R}_+, LG'_0)$ (resp. $E(\mathbf{R}_+, LG'_e)$ if $f(x,y) \in LG'_0$ (resp. LG'_e).*

Proof. The proof follows from the construction which is to follow since the appropriate estimations for the coefficients of $B[u(x,y)](t) = \tilde{u}(t,x) = \sum_{n=0}^{\infty} c_n(y)l_n(t)$ may be easily deduced.

Put

$$B[u(x,y)](t) = \tilde{u}(t,y) = \sum_{n=0}^{\infty} c_n(y) l_n(t), t > 0,$$

(12) $$B[f(x,y)](t) = \sum_{n=0}^{\infty} p_n(y) l_n(t) \text{ where } p_n(y) = \sum_{j=0}^{m} p_{n,j} y^j, n \in \mathrm{N}_0.$$

Note, $a_0(y) = c_0(y)$. Applying the B−transform and by using (1) we transform (11) to

(13) $$-t/4\tilde{u}(t,y) + A(y)\tilde{u}_{yy}(t,y) = \tilde{f}(t,y).$$

In a form of Laguerre series we have

$$-1/4t \sum_{n=0}^{\infty} c_n(y) l_n(t) + A(y) \sum_{n=0}^{\infty} c_n''(y) l_n(t) = 0.$$

Thus, by using

$$-tl_n = (n+1)l_{n+1} - (2n+1)l_n + nl_{n-1} ([6]),$$

it follows

$$\sum_{n=0}^{\infty} A(y) c_n''(y) l_n(t) = 1/4[\sum_{n=1}^{\infty} nc_{n-1}(y) l_n(t) -$$

$$\sum_{n=0}^{\infty} (2n+1) c_n(y) l_n(t) + \sum_{n=0}^{\infty} (n+1) c_{n+1}(y) l_n(t)].$$

This gives the following system of equations

$$4Ac_0'' + c_0 - c_1 = p_0(y),$$

.....

$$4Ac_n'' - nc_{n-1} + (2n+1)c_n - (n+1)c_{n+1} = 4p_n(y), n \in \mathrm{N},$$

(14) .....

Case A. The system (14) becomes

$$4Ac_0'' - c_0 + c_1 = 4(p_{0,0} + p_{0,1} y),$$

.....

$$4Ac_n'' + nc_{n-1} - (2n-1)c_n + (n+1)c_{n+1} = 4(p_{n,0} + p_{n,1}y), n \in \mathbf{N}.$$

Since $c_0'' + c_1'' + \dots + c_n'' = 0$, it follows

$$c_1 = c_0 + 4(p_{0,0} + p_{0,1}y),$$

$$\dots\dots$$

$$c_{n+1} = c_n + 4/(n+1)[\sum_{i=0}^{m} p_{i,0} + \sum_{j=0}^{n} p_{j,1}y], \text{ or}$$

$$c_{n+1} = c_0 + 4[\sum_{t=0}^{n} p_{t,0} \sum_{i=t}^{n} 1/(i+1) + \sum_{t=0}^{n} p_{t,1}y \sum_{i=t}^{n} 1/(i+1)], n \in \mathbf{N_0}.$$

Then,

$$\tilde{u}(t,y) = \sum_{n=0}^{\infty}[c_0 + 4(\sum_{t=0}^{n} p_{t,0} \sum_{i=t}^{n} 1/(i+1) + \sum_{t=0}^{n} p_{t,1}y \sum_{i=t}^{n} 1/(i+1))]l_n(t),$$

and the solution is

$$u(x,y) = \sum_{n=0}^{\infty}\{c_0 + 8\sum_{m=0}^{n-1}(-1)^m \sum_{t=0}^{m-1} p_{t,0}(\sum_{i=t}^{m-1} 1/(i+1))$$

$$+4(-1)^n \sum_{t=0}^{n-1} p_{t,0}(\sum_{i=t}^{n-1} 1/(i+1)) + y[8\sum_{m=0}^{n-1}(-1)^m \sum_{t=0}^{m-1} p_{1,t}(\sum_{i=t}^{m-2}$$

$$1/(i+1)) + 4(-1)^n \sum_{t=0}^{n-1} p_{t,1}(\sum_{i=t}^{n-1} 1/(i+1)))]\}l_n(t).$$

Case B. Summing the system (14) we have

$$(15) \quad 4A(c_0'' + \dots + c_n'') + (n+1)c_n - (n+1)c_{n+1} = 4\sum_{i=0}^{n}\sum_{j=0}^{5} p_{i,j}y^j,$$

or, in ordering form, $(n \in \mathbf{N_0})$,

$$(16) \quad c_{n+1} = c_n - 4A/(n+1)(c_0'' + \dots + c_n'') - 4/(n+1)\sum_{i=0}^{n}\sum_{j=0}^{5} p_{i,j}y^j.$$

Differentiating (15) till the sixth derivative and summing this system we obtain

$$4A(c_0^{(6)}+\dots+c_n^{(6)})+(n+1)c_n^{(4)}-(n+1)c_{n+1}^{(4)}=4\sum_{i=0}^{n}(4!p_{i,4}+5!p_{i,5}y),$$

or

$$(17)\qquad c_{n+1}^{(4)}=c_n^{(4)}-4/(n+1)\sum_{i=0}^{n}(4!p_{i,4}+5!p_{i,5}y), n\in \mathbf{N_0},$$

since

$$c_0^{(6)}+\dots+c_n^{(6)}=0.$$

From (17) we have

$$c_0^{(4)}+\dots+c_n^{(4)}=(n+1)c_0^{(4)}-\sum_{i=0}^{n-1}(4!p_{i,4}+5!p_{i,5}y)$$

$$(18)\qquad \sum_{j=i}^{n-1}4((n-1)+1-j)/(j+1),$$

$$4A(c_0^{(4)}+\dots+c_n^{(4)})+(n+1)c_n^{(2)}-(n+1)c_{n+1}^{(2)}=4\sum_{i=0}^{n}\sum_{j=0}^{3}(5-j)(4-j)p_{i,5-j}y^{(3-j)}.$$

Substituing this in the second derivative of (16), for $n\in \mathbf{N_0}$, we obtain

$$c_0^{(2)}+\dots+c_n^{(2)}=(n+1)c_0^{(2)}+4Ac_0^{(4)}\sum_{j=0}^{n-1}(j+2)(n+1-j)/(j+1)$$

$$-4A\sum_{i=0}^{n-2}(4!p_{i,4}+5!p_{i,5}y)\sum_{s=0}^{n}\sum_{j=i}^{n-2}4A(n+1-s-j)(n+1-j)/(j+1)$$

$$-\sum_{i=0}^{n-1}(\sum_{j=0}^{3}(5-j)(4-j)p_{i,5-j}y^{3-j}))\sum_{j=i}^{n-1}4(n+1-j)/(j+1),(\sum_{i=0}^{n}=0, n\le -1.).$$

Setting this in (16) it follows

$$c_{n+1}=c_n+4A/(n+1)\{(n+2)c_0^{(2)}+4Ac_0^{(4)}\sum_{j=0}^{n-1}(j+2)(n+1-j)/(j+1)-4A\sum_{i=0}^{n-2}$$

$$\sum_{j=0}^{1}(5-j)(4-j)(3-j)(2-j)p_{i,5-j}y^j\sum_{s=0}^{n-2}\sum_{j=i}^{n-2}4(n+1-s-j)(n+1-j)/(j+1)-\sum_{i=0}^{n-1}$$

$$\sum_{j=0}^{3}(5-j)(4-j)p_{i,5-j}y^{3-j}\sum_{j=i}^{n-1}4/(j+1)(n+1-j)\}-4/(n+1)\sum_{i=0}^{n}\sum_{j=0}^{5}p_{i,j}y^j,$$

and setting (12), for $m=5$

$$c_{n+1}=\sum_{i=0}^{5}k_{0,i}y^i+(5!k_{0,5}y+4!k_{0,4})\sum_{i=0}^{n-1}4A/(i+1)\sum_{j=0}^{n-1}4A(j+2)(n-j)/(j+1)+$$

$$\sum_{i=0}^{n-1}\sum_{j=0}^{3}(5-j)(4-j)p_{i,5-j}y^{3-j}\sum_{j=i}^{n-1}4/(j+1)(\sum_{t=0}^{n-1}4(n+1-t)/(t+1))-$$

$$\sum_{i=0}^{n}\sum_{j=0}^{5}p_{i,5-j}y^j\sum_{j=i}^{n}4/(j+1)-\sum_{i=0}^{n-2}(4!p_{i,4}+5!p_{i,5}y)\sum_{j=i}^{n-2}\tag{19}$$

$$\sum_{s=0}^{n-2}16A^2(j+1-s)(j-s)/((j+1)(s+1)),n\in\mathbf{N}.\tag{20}$$

So we have

$$\tilde{u}(t,y)=\sum_{n=0}^{\infty}c_n(y)l_n(t)$$

where $c_n(y)$ is given by (20).

The solution of (11) is the inverse for $\tilde{u}(t,y)$, i. e.

$$B[(\tilde{u}(t,y))](x)=u(x,y)=\sum_{n=0}^{\infty}[(-1)^n c_n+2\sum_{i=0}^{n-1}(-1)^i c_i]l_n(x),([9]).$$

This algorithm we can apply in the case when

$$c_0(y)=\sum_{i=0}^{m}k_{0,i}y^i,p_n(y)=\sum_{i=0}^{m}p_{n,j}y^j,n\in\mathbf{N_0},m\in\mathbf{N_0}.$$

The first step. Differentiating (15) till $(m+1)-$th derivative we get

$$4A(c_0^{(m+1)}+...+c_n^{(m+1)})+(n+1)c_n^{(m-1)}-(n+1)c_{n+1}^{(m-1)}=4\sum_{i=0}^{n}p_i^{(m-1)},$$

and

$$c_{n+1}^{(m-1)} = c_n^{(m-1)} - 4\sum_{i=0}^{n} p_i^{(m-1)}, n \in \mathbf{N_0},$$

since the first part in previous expression is equal to zero. Now, we shall find

$$c_0^{(m-1)} + ... + c_n^{(m-1)} = F(c_0, p_i^{(m-1)}). \tag{21}$$

The second step. Differentiating (15) till $(m-1)$-th derivative , we obtain

$$4A(c_0^{(m-1)}+...+c_n^{(m-1)})+(n+1)c_n^{(m-3)}-(n+1)c_{n+1}^{(m-3)} = 4\sum_{i=0}^{n} p_i^{(m-3)}, n \in \mathbf{N_0},$$

and putting (21) we have

$$c_{n+1}^{(m-3)} = F(c_0, p_i^{(m-1)}, p_i^{(m-3)}), n \in \mathbf{N_0}.$$

The Third step. Repeating this algorithm till the second derivative of c we get:

$$c_0^{(2)} + ... + c_n^{(2)} = F(c_0, p_i^{(m-1)}, p_i^{(m-3)}, ...p_i^{(2)})$$

Setting this in (16) we obtain

$$c_{n+1} = F(c_0, p_i^{(m-1)}, p_i^{(m-3)}, ...p_i^{(2)}, p_i), n \in \mathbf{N_0}.$$

5. Appendix

In the Examples to follow we give the Laguerre evaluation of b, and $B-$transforms for some elements from LG_0, LG_e and their duals. We give also connections between the B and the Henkel and Fouirer transforms.

Example 1.

Since the Hankel transform is defined on LG_0 by

$$\chi_0[\phi](t) = 1/2 < \phi(x), J_0(\sqrt{xt}) >, t \in \mathbf{R}_+, \tag{22}$$

([3], [9], [5]) , it follows

$$\chi_0[\phi](t) = -2b[\phi](t), t \in \mathbf{R}_+, \ .$$

and for $f \in LG_0'(\text{resp.}LG_e'), \phi \in LG_0(\text{resp.}LG_e)$,

$$< B[f(x)](t), \phi(x) >= - < \phi'(t), < f(x), J_0(\sqrt{xt}) >>= -2 < \phi'(t), \chi_0[f](t) >=$$

$$2 < \phi(t), \frac{d}{dt}\chi_0[f](t) > .$$

Thus,

$$B[f(x)](t) = 2\frac{d}{dt}[\chi_0[f(x)](t)], t > 0. \tag{23}$$

Example 2.

$$b[e^{iz\tau}](t) = -2\pi iz \sum_{n=0}^{\infty}(-1)^n \hat{l_n}(z) l_n(t), t \in [0,\infty), Imz > 0,$$

$\hat{l}_n$ is the Fouirer-Laplace transform of $l_n, n \in \mathbf{N_0}$ ([6],[4]). This follows from

$$\hat{l_n}(z) = -(z+1/2)^n/(z-1/2)^{n+1}, n \in N_0, Imz > 0,$$

$$e^{iz\tau} = \sum_{n=0}^{\infty} -(1/2+iz)^n/(-1/2+iz)^{n+1} l_n(\tau), Imz > 0, \tau > 0, ([4]),$$

and

$$b[e^{iz\tau}](t) = -iz < e^{iz\tau}, J_0(\sqrt{t\tau}) >, t > 0.$$

Moreover, the following formula holds

$$b[e^{iz\tau}](t) = e^{-ti/(4z)}, t > 0, Imz > 0.([12], p.41).$$

Example 3.

Let $f \in LG_0'$ (resp. LG_e'). Then ,

$$\hat{f}(-1/(4z)) = (\hat{B}(f))(z), Imz > 0, Rez \in \mathbf{R}.$$

This follows from

$$\hat{B}(f)(z) =< B(f)(s), e^{isz} >=< f(t), b[e^{isz}](t) > .$$

From Example 2. we obtain

$$\hat{B}(f)(z) = < f(t), e^{-ti/(4z)} > = < f(t), e^{itw} > = \hat{f}(w).$$

where $w = -1/(4z)$.

Example 4.

$$b'[J_0(2\sqrt{a\tau})e^{-\tau/2}](t) = (-1/2)e^{-2a}I_0(2\sqrt{at})e^{-t/2}, t > 0, a > 0.$$

It follows from

$$\chi_0[J_0(2\sqrt{a\tau})e^{\tau/2}](t) = e^{2a}I_0(2\sqrt{at})e^{-t/2}([1]).$$

Example 5.

$$b[\phi](t) = -2H[\phi'(r^2)](\rho), t = \rho^2, \tau = r^2, t > 0, \tau > 0, \phi \in LG_0(\text{resp}.LG_e),$$

where

$$H[f](\rho) = < J_0(r\rho), rf(r) >$$

is generalized Hankel transform [1], [12]. This follows from

$$H[\phi'(r^2)](\rho) = < J_0(r\rho), \phi'(r^2)r > = 1/2 < J_0(\sqrt{t\tau}, \phi'(\tau) > = -1/2b[\phi](t).$$

Remark. From

$$b[\varphi](kt)(\tau) = b[\varphi](\tau/k), \tau > 0, k > 0, \varphi \in LG_0(\text{resp}.LG_e)$$

it follows

$$B[f(k\tau)](t) = 1/k^2 B[f](t/k), f \in LG_0'(\text{resp}.LG_e'), t > 0, k > 0.$$

Example 6. Let $f \in LG_0'$ (resp. LG_e'). For $Imp > 0$,

$$B[e^{p\tau}f(\tau)](t) = \frac{d}{dt}[\mathcal{L}(f(\tau)J_0(\sqrt{t\tau}))](p), t > 0, f \in LG_0'(\text{resp}.LG_e').$$

Namely for $\phi \in LG_0($ resp.$LG_e)$, we have

$$< B[e^{p\tau}f(\tau)], \phi(t) > = < e^{p\tau}f(\tau), b[\phi](\tau) > = - < e^{p\tau}f(\tau), < \phi'(t),$$

$$J_0(\sqrt{t\tau}) >>= - < \phi'(t), < e^{p\tau} f(\tau), J_0(\sqrt{t\tau}) >>=$$
$$< \phi(t), < e^{p\tau} f(\tau), J_0'(\sqrt{t\tau}) 1/2\sqrt{\tau/t} >>=$$
$$< \phi(t), \mathcal{L}[f(\tau) 1/2\sqrt{\tau/t}) J_0'(\sqrt{\tau t})](p) > .$$

Particularly ,

$$B[\tau^{-1/2} e^{-p\tau}](t) = -(p^2 + t^2)^{3/2}, t > 0, p > 0.$$

Making the substitution $\tau = r^2, t = \rho^2$ in

$$< B[\tau^{-1/2} e^{-p\tau^{1/2}}], \phi >= - < \phi'(t), < \tau^{-1/2} e^{-p\tau^{1/2}}, J_0(\sqrt{t\tau}) >>$$

we obtain

$$= -4 < \phi'(\rho^2), < r^{-1} e^{-pr} J_0(r\rho) r >>= -4 < \phi'(\rho^2)\rho, H[r^{-1} e^{-pr}](\rho) > .$$

From

$$H[r^{-1} e^{-pr}](\rho) = (p^2 + \rho^2)^{-1/2}([1]),$$

by putting $t = \rho^2$ we continue

$$= -4 < \phi'(\rho^2)\rho, (p^2 + \rho^2)^{-1/2} >= -4 < \phi'(t), 1/2(p^2 + t)^{-1/2} >=$$
$$< \phi(t), (p^2 + t)^{-3/2} > .$$

Example 7.

$$B[e^{-a\tau} J_0(c\sqrt{\tau})](t) = 1/2 e^{-(c^2 - t)/(4a)} [I_0(c\sqrt{t}/(2a)) + c/\sqrt{t} I_0'(c\sqrt{t}/(2a))],$$

$t \in (0, \infty), a > 0, c > 0$, where I_0 is the modified Bessel function of the first kind.

From

$$B[e^{-a\tau} J_0(c\sqrt{\tau})](t), \phi(t) >= - < \phi'(t), < e^{-a\tau} J_0(c\sqrt{\tau}), J_0(\sqrt{t\tau}) >>$$

by the substitution $t = \rho^2, \tau = r^2$ it follows

$$< \phi'(\rho^2)\rho, < e^{-ar^2} J_0(cr) J_0(r\rho) r >>= -4 < \phi'(\rho^2)\rho, H[e^{-ar^2} J_0(cr)](\rho) > .$$

From [1] (p.248, Example 10) we have

$$H[e^{-ar^2}J_0(cr)](\rho) = 1/2a(e^{-(c^2-\rho^2)/(4a)})I_0(c\rho/(2a))$$

$$= 4 < \phi(t), 1/2a[e^{-(c^2-t)/(4a)}(1/(4a))I_0(c\sqrt{t}/(2a)) + e^{-(c^2-t)/(4a)}I_0'(c\sqrt{t}/(2a))$$
$$(c/(4a\sqrt{t})) > = < 2ae^{-(c^2-t)/(4a)}[1/(4a)I_0(c\sqrt{t}/(2a)) +$$
$$I_0'(c\sqrt{t}/(2a))c/(4a\sqrt{t})](t), \phi(t) > .$$

Example 8.

$$B[\mathbf{H}(x)e^{-sx}](t) = 1/(1/2+s)[\sum_{m=0}^{n-1}(-1)^m((-1/2+s)/(1/2+s))^m$$
$$+(-1)^n((-1/2+s)/(1/2+s))^n]l_n(t),$$

where $\mathbf{H}$ is Heaviside function , $s \in C, Re s > 0$. We obtain this from the evaluation of

$$\mathbf{H}(x)e^{-sx} = \sum_{n=0}^{\infty}(s-1/2)^n/(s+1/2)^{n+1}l_n(x), ([11]).$$

Example 9.

Let $m \geq 0$. From [13] we have

$$B[x_+^{m-1}] = B[\Gamma(m)f_m] = \Gamma(m)B[f_m] = \Gamma(m)4^m f_{-(m)} = \Gamma(m)4^m\delta^{(m)}.$$

Finaly,

$$B[x_+^{m-1}](t) = \Gamma(m)2^m\sum_{n=0}^{\infty}[\sum_{i=0}^{m}2^i\binom{m}{i}\binom{n}{i}]l_n(t), t > 0.$$

When $m < 0$ we obtain

$$B[x^{-|m|-1}] = B[\Gamma(-|m|)f_{-|m|}] = \Gamma(-|m|)4^{-|m|}f_{|m|} = \Gamma(-|m|)4^{-|m|}\frac{x^{|m|-1}}{\Gamma(|m|)} =$$

$$\Gamma(-|m|)/\Gamma(|m|)2^{|m|}(|m|-1)!\sum_{n=0}^{\infty}(\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{|m|-1+k}{|m|-1}2^k)l_n(x).$$

(see [11]).

Example 10.

Let $a > 0, k \in \mathbf{N}$. Then,

$$B[f_k(t-a)](x) = 2^k \sum_{n=0}^{\infty}[\sum_{i=0}^{n}\sum_{m=0}^{i-1}\sum_{j=0}^{k}(-1)^m l_m(a) 2^j \binom{k}{j}\binom{n-i-1}{j-1} +$$

$$(-1)^i l_i(a) \sum_{j=0}^{k} 2^j \binom{k}{j}\binom{n-i-1}{j-1}]] l_n(t), n \in N_0. \tag{24}$$

For given $f \in LG'_0(\text{resp.} LG'_e)$ the distribution $f(\cdot - a) \in LG'_0(\text{resp.} LG'_e)$ is defined by

$$< f(t-a), \phi(t) > = < f(t), \phi(t+a) > .$$

Note, $supp f(t-a) \subset [a, \infty)$. By using $(f(t) * \delta(t-a))(x) = f(x-a)$, it follows

$$B[f(t-a)](x) = B[f(t) * \delta(t-a)](x).$$

In our case

$$B[f_k(t-a)](x) = (B[f_k(t)] * B[\delta(t-a)])(x) = 4^k f_{-k}(x) * B[\delta(t-a)](x)$$

$$= 4^k \delta^{(k)}(x) * B[\delta(t-a)](x).$$

Since

$$B[\delta(t-a)](x) = \sum_{n=0}^{\infty}[2\sum_{m=0}^{n-1}(-1)^m l_m(a) + (-1)^n l_n(a)] l_n(x), x > 0,$$

and

$$\delta^{(k)} = \sum_{n=0}^{\infty}[\sum_{i=0}^{k}(1/2)^{k-i}\binom{k}{i}\binom{n}{i}] l_n$$

we obtain (24) using the form of convolution given [9], [11].

We also obtain on $\mathbf{R}_+$

$$\delta^{(m)}(x-a) = \delta^{(m)} * \delta(x-a) = \sum_{n=0}^{\infty}[\sum_{k=0}^{n} l_k(a)$$

$$(\sum_{i=0}^{m}(1/2)^{m-i}\binom{m}{i}\binom{n-k-1}{i-1})] l_n(x).$$

Similarly,

$$(x-a)^m = x_+^m * \delta(x-a) = \sum_{n=0}^{\infty}[\sum_{k=0}^{n} l_k(a)$$

$$(\sum_{i=0}^{m+1} 2^i \binom{m+1}{i}\binom{n-k-1}{i-1})]l_n(x).$$

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REZIME

PRIMENA UOPŠTENE B-TRANSFORMACIJE

Razmatrana je jednačina $xu_{xx}(x,y)+2u_x(x,y)+Au_{yy}(x,y) = f(x,y), f(x,y) = \sum_{n=0}^{\infty} a_n(y)l_n \in E(\mathbf{R}_+, LG'_0)$ (resp. $E(\mathbf{R}_+, LG'_e)$), $A \in \mathbf{R}$. Rešavamo odgovarajući granični problem u $E(\mathbf{R}_+, LG'_e))$ (Prop. 1). Nalazimo tešenje u obliku Lagerovog reda ako je A glatka funkcija i f ima pogodnu ekspanziju po x (Prop.2). Ako je A glatka funkcija na $(0,\infty)$ nalazimo rešenje ove jednačine u $E(\mathbf{R}_+, LG'_0)$ (resp. $E(\mathbf{R}_+, LG'_e)$), pretpostavljajući da a_n, $n \in \mathrm{N}_0$, i da je prvi koeficijent rešenja u obliku Lagerovog reda polinom proizvoljnog ali fiksnog stepena (resp. prvog stepena).

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