Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 24, 1 (1994), 129-137

Review of Research
Faculty of Science
Mathematics Series

# THE LEBESGUE DECOMPOSITION OF THE NULL-ADDITIVE FUZZY MEASURES

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**Abstract**

In this paper the connection between two type of absolute continuity of a fuzzy measure $m$ with respect to a given null-additive fuzzy measure $g$ is investigated. Two theorems of Lebesgue decomposition type for null-additive fuzzy measures are proved.

*AMS Mathematics Subject Classification (1991): 28A10, 28E10*
*Key words and phrases:* null-additive fuzzy measure, continuous from above, continuous from below, autocontinuous from above, autocontinuous from below.

## 1. Introduction

Wang [25] has introduced the notion of the null-additive set function $m$, i.e. such that $m(B) = 0$ implies $m(A \cup B) = m(A)$ for $A \cap B = \emptyset$. This property of set functions was noticed earlier by L.Drewnowski [6] as a part of the investigations of special class of set functions, which was introduced by I.Dobrakov [4].Recently there were published many papers on null-additive set functions: H.Suzuki [23],[24], E.Pap [18],[19],[20] and Z.Wang [26].It

[1]This research was supported by Science Fund of Serbia, grant number 0401A,through Matematički institut

tourns out that many important generaly non-additive set functions are included in this class of set functions as t-conorm decomposable measures (E.Pap [14], [15], [16], S. Weber [27]), pseudo-additive measures (H. Ichihashi, M. Tanaka, K. Asai [9], T. Murofushi, M. Sugeno [11]), k-triangular set functions (E. Pap [12], [13], E. Guariglia [7]), etc..
In this paper we shall prove the connection between two type of absolute continuity of a fuzzy measure $m$ with respect to a given null-additive fuzzy measure $g$. We have proved in papers [15] and [17] Lebesgue decomposition theorems for decomposable measures. Now we shall prove two theorems of Lebesgue decomposition type .

## 2. Null-additive fuzzy measures

Throughout this paper $\Sigma$ always denotes a $\sigma$-ring of subsets of the given set $X$.

**Definition 1.** *A set function $m$, $m : \Sigma \to [0, \infty]$, is called null-additive, if we have*

$$m(A \cup B) = m(A)$$

*whenever $A, B \in \Sigma$, $A \cap B = \emptyset$, and $m(B) = 0$.*

**Definition 2.** *A fuzzy measure $m, m : \Sigma \to [0, \infty]$, is a nonnegative extended real - valued set function $m$ defined on $\sigma$ - ring $\Sigma$ and with the properties:*

$$(FM_1) \qquad m(\emptyset) = 0,$$

$$(FM_2) \qquad E \subset F \quad \Rightarrow \quad m(E) \leq m(F).$$

For fuzzy measures we do not need the condition "$A \cap B = \emptyset$" in Definition 1. In some papers ([23], [25], [26]) fuzzy measures have two continuity properties more:

**Definition 3.** *A fuzzy measure $m, m : \Sigma \to [0, \infty]$, is continuous from below if it satisfies the condition*

$$(FM_3) \qquad E_1 \subset E_2 \subset ... \quad , E_n \in \Sigma \quad \Rightarrow \quad m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} m(E_n).$$

**Definition 4.** *A fuzzy measure $m, m : \Sigma \to [0, \infty]$, is continuous from above if it satisfies the condition*

$$(FM_4) \qquad E_1 \supset E_2 \supset \dots \qquad , E_n \in \Sigma$$
$$\text{and there exists } n_0 \text{ such that } m(E_{n_0}) < \infty \Rightarrow \mu(\textstyle\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} m(E_n).$$

We have by [18]

**Definition 5.** *A set function $m$ is called autocontinuous from above (resp. from below) if for every $\epsilon > 0$ and every $A \in \Sigma$, there exists $\delta = \delta(A, \epsilon) > 0$ such that*

$$m(A) - \epsilon \le m(A \cup B) \le m(A) + \epsilon \ (\textit{resp. } m(A) - \epsilon \le m(A \setminus B) \le m(A) + \epsilon)$$

*whenever $B \in \Sigma, A \cap B = \emptyset$ (resp. $B \subset A$) and $m(B) < \delta$ holds.*

By Proposition 3. from [25] any set function which is autocontinuous from above (below) is null-additive.

## 3. Absolute continuity with respect to a fuzzy measure

**Definition 6.** *Let $m$ and $g$ be two finite fuzzy measures. If $E \in \Sigma, g(E) = 0$ implies $m(E) = 0$, then we say that $m$ is absolutely continuous with respect to $g$.*

**Definition 7.** *Let $m$ and $g$ be two finite fuzzy measures. If for every $\epsilon > 0$ there is a $\delta > 0$ such that $E \in \Sigma, g(E) < \delta$ implies $m(E) < \epsilon$, then we say that $m$ is absolutely $\epsilon-$continuous with respect to $g$.*

**Theorem 1.** *Let $m$ and $g$ be two finite fuzzy measures such that they are continuous from above and continuous from below. If $g$ is autocontinuous from above, then $m$ is absolutely continuous with respect to $g$ iff $m$ is absolutely $\epsilon-$continuous with respect to $g$.*

*Proof.* It is obvious that if $m$ is absolutely $\epsilon-$ continuous with respect to $g$, then $m$ is absolutely continuous with respect to $g$.

Suppose now that $E \in \Sigma, g(E) = 0$ implies $m(E) = 0$. If the theorem would not be true, then there would exist $\epsilon > 0$ and a sequence $\{E_n\}$ from $\Sigma$ such that

$$g(E_n) < \frac{1}{n} \quad \textit{and} \quad m(E_n) > \epsilon \quad (n \in N). \tag{1}$$

Since $g$ is autocontinuous from above there exists a subsequence $\{E_{n_k}\}$ of the sequnce $\{E_n\}$ such that

$$g(\bigcup_{i=s}^{k} E_{n_i}) < \frac{1}{s} \quad \textit{for} \quad s = 1, 2, ..., k. \tag{2}$$

By the continuity from above of $g$ we have

$$\lim_{s \to \infty} g(\bigcup_{i=s}^{\infty} E_{n_i}) = g(\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} E_{n_i}). \tag{3}$$

Since $g$ is continuous from below we obtain by (2)

$$g(\bigcup_{i=s}^{\infty} E_{n_i}) = \lim_{k \to \infty} g(\bigcup_{i=s}^{k} E_{n_i}) \leq \frac{1}{s}.$$

Hence by (3)

$$g(\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} E_{n_i}) = 0,$$

which implies

$$m(\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} E_{n_i}) = 0.$$

On the other hand, we obtain by the continuity from above and continuity from below of the fuzzy measure $m$ and (1)

$$m(\bigcap_{s=1}^{\infty} \bigcup_{i=s}^{\infty} E_{n_i}) = \lim_{s \to \infty} m(\bigcup_{i=s}^{\infty} E_{n_i}) = \lim_{s \to \infty} \lim_{k \to \infty} m(\bigcup_{i=s}^{k} E_{n_i}) \geq m(E_{n_p}) > \epsilon.$$

Contradiction.

**Theorem 2.** *Let $m$ be a finite null-additive fuzzy measure which is continuous from above and continuous from below. Then there exists a set $A$ from $\Sigma$ such that*

$$m(A) = \sup\{m(E), E \in \Sigma\}, \tag{4}$$

$$m(E \setminus A) = 0 \text{ and } m(E) = m(E \cap A) \quad (E \in \Sigma).$$

*Proof.* We shall choose a sequence $\{A_n\}$ from $\Sigma$, which will generate the desired set $A$. Let $A_0 = \emptyset$. We take $A_1$ from $\Sigma$ such that

$$m(A_1) = \sup\{m(E) : E \in \Sigma\}.$$

This is possible by the continuity from below of $m$. We choose $A_2$ from $\Sigma$ such that

$$m(A_2) = \sup\{m(E) : E \subset X \setminus A_1\}.$$

Repeating this procedure, we choose a sequence $\{A_n\}$ such that

$$m(A_n) = \sup\{m(E) : E \subset X \setminus \bigcup_{i=0}^{n-1} A_i, E \in \Sigma\} \tag{5}$$

holds. We take $A = \bigcup_{i=0}^{\infty} A_i$. Then by the construction (4) holds. The continuity from above of $m$ implies

$$\lim_{n\to\infty} m(E \setminus \bigcup_{i=0}^{n} A_i) = m(E \setminus A). \tag{6}$$

By (5) we obtain

$$\limsup_{n\to\infty} m(A_n) \geq \lim_{n\to\infty} m(E \setminus \bigcup_{i=0}^{n} A_i).$$

Hence by the exhaustivity of $m$ (Proposition 1, [18]) and (6) $m(E \setminus A) = 0$. Hence by the null-additivity of $m$

$$m(E) = m((E \cap A) \cup (E \setminus A)) = m(E \cap A).$$

## 4. Lebesgue decomposition

**Definition 8.** *Let $m$ and $g$ be two finite fuzzy measures defined on $\Sigma$. The fuzzy measure $m$ is called singular with respect to $g, m \perp g$, if there exists a set $A$ from $\Sigma$ such that*

$$m(E \setminus A) = g(E) = 0 \;\; (E \in \Sigma).$$

**Remark.** By Theorem 2. if for null-additive fuzzy measures $m$ and $g$, which are continuous from above and continuous from below, $m \perp g$ holds, then we have $g \perp m$ too.

Now we have the following two theorems of Lebesgue decomposition type.

**Theorem 3.** *Let $m$ and $g$ be two finite null-additive fuzzy measures on $\Sigma$. Then there exist two null-additive fuzzy measures $m_c$ and $m_s$ such that $m_c(E) = m(E \setminus A)$ and $m_s(E) = m(E \cap A)$ for a set $A \in \Sigma$ and $m_c$ is absolutely continuous with respect to $g$ and $m_s$ is singular with respect to $g$.*

*Proof.* The family

$$\Sigma_1 = \{E \in \Sigma : g(E) = 0\}$$

is a $\sigma-$ subring of the $\sigma-$ ring $\Sigma$. By Theorem 2. the restriction of $m$ on $\Sigma_1$ has a set $A \in \Sigma_1$ such that $m(E \setminus A) = 0$ and $m(E) = m(E \cap A)$ for $E \in \Sigma_1$. We take

$$m_c(E) = m(E \setminus A)$$

and

$$m_s(E) = m(E \cap A)$$

for each $E \in \Sigma$. It is easy to check that $m_c$ and $m_s$ are null-additive fuzzy measures and that $m_c$ is absolutely continuous with respect to $g$ and $m_s$ is singular with respect to $g$.

**Theorem 4.** *Let $m$ and $g$ be two finite autocontinuous from above fuzzy measures on $\Sigma$, which are continuous from above and from below. Then* *there exist two autocontinuous from above fuzzy measures $m_c$ and $m_s$ such that $m_c(E) = m(E \setminus A)$ and $m_s(E) = m(E \cap A)$ for a set $A \in \Sigma$ and $m_c$ is absolutely $\epsilon-$ continuous with respect to $g$ and $m_s$ is singular with respect to $g$.*

*Proof.* We take same $m_c$ and $m_s$ as in the proof of Theorem 3. Then by Theorem 1. $m_c$ is absolutely $\epsilon-$ continuous.

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**REZIME**

TEOREMA O LEBESGUEOVOJ DEKOMPOZICIJI ZA NULA-ADITIVNE FAZI MERE

Ispituje se veza izmedju dve vrste apsolutne neprekidnosti nula-aditivne fazi mere $m$ u odnosu na drugu fazi meru $g$. Dokazuju se dve teoreme tipa Lebesguove dekompozicije za nula-aditivne fazi mere.

*Received by the editors May 4, 1992*