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A COMPARISON OF NONLINEAR AND LINEAR LOCAL PREDICTIONS FOR THE STOCHASTIC PROCESS

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Abstract

The mean square error $\hat{\delta}(s)(\bar{\delta}(s))$ of a nonlinear (linear) local prediction of $X(t_0)$ by $\{X(u), u \leq s\}, s < t_0,$ is defined as the infinitesimal $\delta(s), s \to t_0$. The comparison of $\hat{\delta}(s)$ and $\bar{\delta}(s)$ is discussed.

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Let $\{X(t), t > 0\}$ be a real second order process, left-continuous at a point t_0. The nonlinear prediction $\hat{X}(s)$ of $X(t_0)$ by $\{X(u), u \leq s\}, s < t_0$, is the conditional expectation $\hat{X}(s) = E_s X(t_0) = E(X(t_0)|X(u), u \leq s)$. Denote by $\hat{\delta}(s) = \|X(t_0) - \hat{X}(s)\|^2 = E(X(t_0) - \hat{X}(s))^2$ the mean square error of this prediction. The linear prediction $\bar{X}(s)$ of $X(t_0)$ by $\{X(u), u \leq s\}$ is the projection of $X(t_0)$ onto the mean square linear closure of $\{X(u), u \leq s\}$. Denote by $\bar{\delta}(s) = \|X(t_0) - \bar{X}(s)\|^2$ the mean square error of this prediction. Evidently $\hat{\delta}(s) \leq \bar{\delta}(s)$. We shall mention here that the class of process for which $\bar{X}(s) = \hat{X}(s)$ for all $t_0 > 0$ is larger than the class of Gaussian processes (see [3]).

Example. Let $X(t) = w^3(t)$, where $\{w(t), t > 0\}$ is a standard Wiener process. We write $X(t) = H_3(w(t)) + 3tw(t)$ introducing the one-dimensional

Hermite polynomials $H_p(\xi)$ of the degree $p = 1, 2, \ldots$ in the Gaussian variable ξ. It is easy to find $\hat{X}(s) = H_3(w(s)) + 3tw(s)$ as $\{H_p(w(t)), t > 0\}$ is a martingale. So $\hat{\delta}(s) = \|X(t_0)\|^2 - \|\hat{X}(s)\|^2 = (3!t_0^3 - 9t_0^2 \cdot t_0) - (3!s^3 - 9t_0 \cdot s) = 15t_0^3 - 6s^3 - 9t_0^2 s$ or, developing $\hat{\delta}(s)$ in the Taylor series $\hat{\delta}(s) = 27t_0^2(t_0 - s) - 18t_0(t_0 - s)^2 + 6(t_0 - s)^3$.

Conserning the linear prediction $\bar{X}(s)$ we are going to find the integral representation $X(t) = \int_0^t g(t,x)dZ(x), \{Z(t), t > 0\}$ is a wide sense martingal, and to show that this representation is a proper canonical one. Then $\bar{\delta}(s) = \int_s^{t_0} g^2(t_0, x)dF(x),\ F(x) = \|Z(x)\|^2$. Starting from the correlation function $R(s,t) = EX(s)X(t) = 9ts^2 + 6s^3, s \le t,$ we seek the integral representation in the form

$$X(t) = \int_0^t (at + bx)dZ(x), \quad \|dZ(x)\|^2 = dx. \tag{1}$$

From $R(s,t) = \int_0^s (at + bx)(as + bx)dx = 9ts^2 + 6s^3, s \le t,$
we obtain

$$\left.\begin{array}{r} 2a^2 + ab = 18 \\ 2b^2 + 3ab = 36 \end{array}\right\} \text{ and} \tag{2}$$

$$a = 3/2(\sqrt{11} - \sqrt{3}) = 2,38\ldots, b = 3/2(3\sqrt{3} - \sqrt{11}) = 2,82\ldots$$

The process $\{X(t)\}$ belongs to the class of processes considered in [1] because $a + b = \sqrt{27} \ne 0$. If follows that (1) with (2) is the proper canonical representation. Finally, $\bar{\delta}(s) = (a+b)^2 t_0^2(t_0 - s) - b(a+b)t_0(t_0 - s)^2 + 1/3b^2(t_0 - s)^3 = 27t_0^2(t_0 - s) - 14,65\ldots t_0(t_0 - s)^2 + 2,65\ldots(t_0 - s)^3$.

We define the mean square error of the nonlinear (linear) local prediction as the infinitesimal $\hat{\delta}(s)(\bar{\delta}(s))$ as $s \to t_0$. The comparison of the nonlinear local and linear local predictions is the comparison of the infinitesimals $\hat{\delta}(s)$ and $\bar{\delta}(s)$. For instance, the nonlinear and linear local predictions in the above example are equivalent.

Proposition 1. *All the relations between the nonlinear and linear local predictions are posible, except the one that the nonlinear is of a higher order than the linear.*

Proof. Let $\xi_1, \xi_2, \ldots$ be a sequence of a Gaussian variables, $E\xi_k = 0, \|\xi_k\|^2 = 1$. Let $a_1, a_2, \ldots$ be a sequence of numbers such that

$\sum_1^\infty k! a_k^2 < \infty$. Finally, let $t_0 > 0$ be fixed. We define the process $\{X(t), t > 0\}$ by

$$X(t) = \begin{cases} \sum_{k=1}^{n(t)} a_k H_k(\xi_k) & \text{if } (1 - 1/2^{n-1})t_0 < t \le (1 - 1/2^n)t_0 \\ \sum_{k=1}^{\infty} a_k H_k(\xi_k) & \text{if } t = t_0 \\ \text{arbitrary} & \text{if } t > t_0. \end{cases}$$

It follows from the definition that $\{X(t)\}$ is left-contionuous at $t = t_0$. Also, since $\{X(t)\}$ is a wide sense martingale, $\bar{X}(s) = X(s) = \sum_{k=1}^{n(s)} a_k H_k(\xi_k)$ with $\bar{\delta}(s) = \| \sum_{k=n(s)+1}^{\infty} a_k H_k(\xi_k)\|^2 = \sum_{k=n(s)+1}^{\infty} a_k^2 k!$ $(\|H_p(\xi)\|^2 = k! \|\xi\|^2)$. To find $\hat{X}(s)$ we use the comutativity $E(H_p(\xi)|\xi_1, \xi_2, \ldots) = H_p(E(\xi|\xi_1, \xi_2, \ldots))$, [2], and the smoothing property of the conditional expectation. We have $\hat{X}(s) = E_s(X(s) + \sum_{k=n(s)+1}^{\infty} a_k H_k(\xi_k))m = X(s) + \sum_{k=n(s)+1}^{\infty} a_k E_s H_k(\xi_k) = X(s) + \sum_{k=n(s)+1}^{\infty} a_k H_k(E(\xi_k|\xi_1, \ldots, \xi_{n(s)}))$ and $\hat{\delta}(s) = \sum_{k=n(s)+1}^{\infty} a_k^2 \|H_k(\xi_k) - H_k \bar{\xi}_k)\|^2 = \sum_{k=n(s)+1}^{\infty} a_k^2 k! (\|\xi_k\|^{2k} - \|\bar{\xi}_k\|^{2k})$, $\bar{\xi}_k = E(\xi_k|\xi_1, \ldots, \xi_{n(s)})$.

Mention that $\|\bar{\xi}_k\|^2 \le \|\xi_k\|^2 = 1$ or $\hat{\delta}(s) \le \bar{\delta}(s)$.

For a given $a_1, a_2, \ldots$ the function $\hat{\delta}(s), s \le t_0$, depends on the distribution of $\xi_1, \xi_2, \ldots$. For example, if $\xi_1, \xi_2, \ldots$ are independent, then $\bar{\xi}_k = D$ and $\hat{\delta}(s) = \bar{\delta}(s)$; if $\xi_1 = \xi_2 = \ldots$ then $\bar{\xi}_k = \xi_1$ and $\hat{X}(s) = X(t_0)$ or $\hat{\delta}(s) = 0$.

Choosing the appropriate distribution of $\xi_1, \xi_2, \ldots$ we obtain that the infinitesimal $\bar{\delta}(s)$ is equivalent, of the same order, of a higher order or incommensurable to the infinitesimal $\hat{\delta}(s)$ □.

Finally we shall consider a class of processes with equivalent nonlinear and linear local predictions for all $t_0 > 0$. Let $\{\eta(t), t > 0\}$ be a Gaussian martingale with a $\|\eta(t)\|^2 = F(t)$ continuous function. Then $\{H_p(\eta(t)), t > 0\}, p = 1, 2, \ldots$ is the martingale with $\|H_p(\eta(t))\|^2 = p! F^p(t)$. Let $0 < k_i, \le p, i = 1, \ldots m, p > 1$. The process $\{Z(t), t > 0\}$ defined by

$$Z(t) = \sum_i F^{p-k_i}(t) H_{k_i}(t) / \sum_j k_j! F^{p-k_j}(t), \ H_{k_i}(t) = H_{k_i}(\eta(t))$$

is a wide sense martingale: for $s \le t, < Z(s), Z(t) >=$ $[(\sum_j k_j! F^{p-k_i}(s))(\sum_j k_j F^{p-k_j}(t))]^{-1} \cdot \sum_i F^{p-k_i}(s) F^{p-k_i}(t) k_i! F^{k_i}(s) =$ $F^p(s) / \sum_j k_j! F^{p-k_j}(s) = \|Z(s)\|^2 = F_1(s)$.

But, since $E_s Z(t) = \sum_i F^{p-k_i}(t) H_{k_i}(s) / \sum_j k_j! F^{p-k_j}(t) \neq Z(s), s < t$, is not a martingale.

Starting from $\{Z(t)\}$ as an innovation process, we consider the process $\{X(t), t > 0\}$ with the proper canonical representation $X(t) = \int_0^t g(t,x)dZ(x)$. The process $\{X(t)\}$ is a curve in the Hilbert space $\mathcal{H}$-the linear closure of all the one-dimensional polynomials $\{P_n(\eta(t)), t > 0, n = 1, \ldots, k_m\}$.

Suppose that the response function $g(t,x), 0 \le x \le t$ is continuous and that $g(t,t) > 0$ for all $t > 0$.
Then the infinitesimal $\bar{\delta}(s) = \int_s^{t_0} g^2(t_0,x)dF_1(x)$ is of the order $dF(t_0)$, $dF_1(t) = d(F^p(t)/\sum_j k_j!F^{p-k_j}(t))$. From $\hat{X}(s) = \bar{X}(s) + E_s\int_s^{t_0} g(t_0,x)dZ(x)$, it follows that $\hat{X}(s) - \bar{X}(s) = \int_s^{t_0} g(t_0,x)E_s dZ(x)$. But $E_s(Z(x+h) - Z(x)) = \sum_i [F^{p-k_i}(x+h)/\sum_j k_j!F^{p-k_j}(x+h) - [F^{p-k_i}(x)/\sum_j k_j!F^{p-k_j}(x)]H_{k_i}(s)$ or $E_s dZ(x) = \sum_i d(F^{p-k_i}(x)/\sum_j F^{p-k_j}(x)k_j!)H_{k_i}(s) = (\sum_i \Phi_i(x)H_{k_i}(s))dF(x)$, where $\Phi_i(x)$ is a non-random function. Consider the infinitesimal

$$\bar{\delta}(s) - \hat{\delta}(s) = \|\hat{X}(s) - \bar{x}(s)\|^2 = \|\sum_i H_{k_i}(s)\int_s^{t_0} g(t_0,x)\Phi_i(x)dF(x)\|^2 \sim$$

$$\sim (\sum_i k_i!F^{k_i}(t_0)g^2(t_0,t_0)\Phi_i^2(t))(dF(t_0))^2.$$

We conclude that $\bar{\delta}(s)$ and $\hat{\delta}(s)$ are equivalent.

References

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REZIME

UPOREDJIVANJE NELINEARNOG I LINEARNOG LOKALNOG PREDVIDJANJA SLUČAJNOG PROCESA

Definiše se srednje kvadratna greška $\hat{\delta}(s)(\bar{\delta}(s))$ nelinearnog (linearnog) lokalnog predvidjanja za $X(t_0)$ pomoću $\{X(u), u \leq s\}, s < t_0$ kao beskonačno mala $\delta(s), s \to t_0$. Razmatra se odnos $\hat{\delta}(s)$ i $\bar{\delta}(s)$.

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