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THE ERROR OF APPROXIMATION FOR A PARTICULAR SOLUTION

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Abstract

In this paper a measure of approximation is constructed for the approximate solution of the nonhomogeneous differential equation which had been constructed in paper [5]. The measures of approximation in the spaces $\mathcal{F}_0$, L and C do not depend on the length of the interval $[0, T]$ (on the t axis).

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1. Introduction

In the field of Mikusinski operators, let us observe the nonhomogeneous differential equation

$$\sum_{\mu=0}^{m}\sum_{k=0}^{n}\alpha_{\mu,k}s^k x^{(\mu)}(\lambda) = f(\lambda), \ \ 0 \leq \lambda \leq \lambda_1 \tag{1}$$

with conditions

$$x^{(\mu)}(0) = \varphi_\mu, \ \ \mu = 0, 1, \dots, m-1, \tag{2}$$

where

$$f(\lambda) = \{f_1(\lambda,t)\} + \sum_{\mu=0}^{m}\sum_{k=1}^{n}\sum_{\nu=0}^{k-1} s^{n-\nu-1}\alpha_{\mu,n-k+\nu}\frac{\partial^{\mu+k}x(\lambda,0)}{\partial\lambda^\mu\partial t^k}, \tag{3}$$

s is the differential operator, $f(\lambda)$ is a continuous operator function and $\alpha_{\mu,\nu}$ are numerical constants. Equation (1) corresponds to the nonhomogeneous partial differential equation with constant coefficients ([2])

$$\sum_{\mu=0}^{m}\sum_{k=0}^{n}\alpha_{\mu,k}\frac{\partial^{\mu+k}x(\lambda,t)}{\partial\lambda^\mu\partial t^k} = f_1(\lambda,t),\ \ 0\le\lambda\le\lambda_1,\ \ 0\le t\le\infty, \tag{4}$$

with conditions

$$\frac{\partial^{\mu+k}x(\lambda,0)}{\partial\lambda^\mu\partial t^k} = \psi_{\mu,k}(\lambda)\ \text{ for }\ \mu=0,\dots,m,\ \ k=0,\dots,n-1, \tag{5}$$

$$\frac{\partial^{\mu}x(0,t)}{\partial\lambda^\mu} = \varphi_\mu(t)\ \text{ for }\ \mu=0,\dots,m-1. \tag{6}$$

The particular solution of equation (1) is of the form ([5]):

$$x_p(\lambda) = \sum_{j=1}^{m} A_j\int_0^\lambda \exp((\lambda-x)\omega_j)f(x)dx, \tag{7}$$

where

$$A_j = \frac{I}{P'(\omega_j)} \tag{8}$$

and ω_j are simple solutions of the characteristic equation of equation (1) which are logarithmic and can be written as

$$\omega_j = \sum_{i=0}^{\infty} c_{i,j}l^{i\alpha_j-\beta_j},\ \ \alpha_j>0,\ \beta_j\le 1,\ \ j=1,\dots,m. \tag{9}$$

The approximate particular solution of equation (1) can be written as ([5])

$$x_{p,n}(\lambda) = \sum_{j=1}^{m} A_{j,n}\int_0^\lambda \exp((\lambda-x)\omega_{j,n})f(x)dx, \tag{10}$$

where

$$A_{j,n} = \frac{I}{P'(\omega_{j,n})} \tag{11}$$

and $\omega_{j,n}$ are the approximate solutions of the characteristic equation, so that

$$\omega_{j,n} = \sum_{i=0}^{n} c_{i,j} l^{i\alpha_j - \beta_j}. \tag{12}$$

2. Preliminary notations and notions

The convergence in L (L is the space of locally integrable functions) is the convergence in all seminorms of the form

$$||f||_T = \int_0^T |f(t)|dt, \;\; T > 0, \;\; f \in L.$$

$\mathcal{F}_0$ is the algebra of all operators of the form f/g $f \in L$ and $g \in L_0$, where L_0 is the subspace of L consisting of all functions f such that $||f||_T > 0$, for every $T > 0$. J. Burzyk introduced the functional

$$B_{T,\varepsilon}(x) = \inf\{||f||_T : \;\; x = \frac{f}{g}, \; ||g||_T < 1, \;\; ||l - lg||_T < \varepsilon\} \tag{13}$$

for $x \in \mathcal{F}_0$.

A sequence $\{x_n\}$ converges type I' to x; x_n, $x \in \mathcal{F}_0$, iff $B_{T,\varepsilon}(x_n - x) \to 0$ as $n \to \infty$ for any $T, \varepsilon > 0$. ([1])

The convergence type I' is equivalent to the convergence defined by $A(x)$ where

$$A(x) = \sum_{i=0}^{\infty} \frac{B_{i,\frac{1}{i}}(x)}{e^{ie^{i^2}}(1 + B_{i,\frac{1}{i}}(x))}, \;\; \text{for } x \in \mathcal{F}_0. \tag{14}$$

Definition 1. *([4]) Operator $\tilde{x} \in \mathcal{F}_0$ is the approximation of the operator $x \in \mathcal{F}_0$, according to the functional $A(x)$ with the measure of approximation $\delta > 0$, if $A(x - \tilde{x}) < \delta$.*

Definition 2. *([4]) Function $\tilde{f}$ from L is the approximation of function f from L according to the functional*

$$F(f) = \sum_{i=1}^{\infty} \frac{1}{e^{ie^{i^2}}} \frac{||f||_i}{1 + ||f||_i}, \;\; f \in L \tag{15}$$

with the measure of approximation $\delta_L > 0$ if $F(f - \tilde{f}) < \delta_L$.

3. Estimations in the space $\mathcal{F}_0$

Let us suppose that the operator

$$\int_0^\lambda \sum_{j=1}^m A_j f(x) g \exp((\lambda - x)\omega_j) dx$$

for $g \in L_0$ represents the continuous function, then operators

$$y_n(\lambda) = \frac{\int_0^\lambda (\sum_{j=1}^m A_{j,n} f(x) g \exp((\lambda - x)\omega_{j,n}) dx}{g}$$

(16)

$$y(\lambda) = \frac{\int_0^\lambda (\sum_{j=1}^m A_j f(x) g \exp((\lambda - x)\omega_j) dx}{g}$$

belong to the space $\mathcal{F}_0$.

In order to estimate the functional $A(y_n(\lambda) - y(\lambda))$, let us observe

$$(17) \qquad \begin{aligned} B_{T,\varepsilon}(y_n(\lambda) - y(\lambda)) = \ & \inf\{||J_g(\lambda,t)||_T,\ g \in L_0, \\ & ||g||_T < 1,\ ||l - lg||_T < \varepsilon\} \end{aligned}$$

where

$$\frac{\int_0^\lambda f(x) g \sum_{j=1}^m (A_{j,n} \exp((\lambda - x)\omega_{j,n}) - A_j \exp((\lambda - x)\omega_j)) dx}{g} =: \frac{\{J_g(\lambda,t)\}}{g}.$$

The operator $g_1 = \frac{lk}{I+kl}$ satisfies the inequalities $||g_1||_T < 1$, $||l - lg_1||_T < \frac{1}{k}$, for every $T > 0$ and $k > 0$. From relation (17) follows

$$B_{t,\varepsilon}(y_n(\lambda) - y(\lambda)) \le ||J_{g_1}(\lambda,t)||_T.$$

In order to estimate $||J_{g_1}(\lambda,t)||_T$ let us introduce

$$\{J_{g_1}^1(\lambda,t)\} := \int_0^\lambda \sum_{j=1}^m f(x) g_1 A_{j,n} (\exp(\lambda - x)\omega_{j,n} - \exp(\lambda - x)\omega_j) dx$$

$$(18) \qquad \{J_{g_1}^2(\lambda,t)\} := \int_0^\lambda \sum_{j=1}^m f(x) g_1 (A_{j,n} - A_j) \exp((\lambda - x)\omega_j) dx.$$

Supposing that $f_{2,j}(x,t)$, where $f(x)A_{j,n} = \{f_{2,j}(x,t)\}$), represent functions from L by t and continuous by x, then there exist $\tilde{A}_j(\lambda, T)$ such that

$$\tilde{A}_j(\lambda,T) = \max_{0\le x\le\lambda} \int_0^T |f_{2,j}(x,t)|dt,\ j = 1,\dots,m. \tag{19}$$

The following estimation

$$||g_1(\exp(\lambda - x)\omega_{j,n} - \exp(\lambda - x)\omega_j)||_T \le \frac{k_{g_1}(\lambda - x, T, \alpha_j, \beta_j)}{\Gamma(\frac{(n+1)\alpha_j - \beta_j - 1}{2} + 1)}$$

can be obtained analogously as in paper [3].

Denoting by

$$K_{g_1}^M(\lambda, T, \alpha_j, \beta_j) = \max_{0\le x\le\lambda} k_{g_1}(\lambda - x, T, \alpha_j, \beta_j), \tag{20}$$

we have

$$\int_0^T |J_{g_1}^1(\lambda,t)|dt \le \lambda \sum_{j=1}^m \tilde{A}_j(\lambda,T) \frac{K_{g_1}^M(\lambda, T, \alpha_j, \beta_j)}{\Gamma(\frac{(n+1)\alpha_j - \beta_j - 1}{2} + 1)}. \tag{21}$$

The estimation for the difference $|A_{j,n} - A_j|$ is obtained as

$$|A_{j,n} - A_j| \le \frac{l^{\nu_1 + \beta_j(\mu_1 - 1)}}{\mu_1 |\alpha_{\mu_1,\nu_1}||c_{\mu_1}|} \cdot \sum_{k=1}^{\infty} \frac{(-1)^k}{|c_{\mu_1}|^k} \left(|l^{\alpha_j}(W_{j,\mu_1,n} - W_{j,\mu_1})| + \right.$$

$$\left. + |L_{j,n} - L_j|\right) \left(\sum_{i=0}^{k-1} |l^{\alpha_j} W_{j,\mu_1,n} + L_{j,n}|^i |l^{\alpha_j} W_{j,\mu_1} + L_j|^{k-1-i} \right),$$

where W_{j,μ_1} and $W_{j,\mu_1,n}$ are obtained from:

$$l^{\alpha_j} W_{j,\mu} + c_\mu I = (l^{\beta_j}\omega_j)^{\mu-1};\ \ l^{\alpha_j} W_{j,\mu,n} + c_\mu I = (l^{\beta_j}\omega_{j,n})^{\mu-1}$$

and α_j, β_j are given in relation (9) and

$$L_j = \sum_{\mu=1}^{m}{}^* \sum_{k=0}^{n}{}^* \frac{\mu\alpha_{\mu,k}}{\mu_1 \alpha_{\mu_1,\nu_1}} l^{\nu_1 - k + \beta_j(\mu_1 - \mu)} (l^{\beta_j}\omega_j)^{\mu-1},\ j = 1,\dots,m$$

$$L_{j,n} = \sum_{\mu=1}^{m}{}^* \sum_{k=0}^{n}{}^* \frac{\mu\alpha_{\mu,k}}{\mu_1 \alpha_{\mu_1,\nu_1}} l^{\nu_1 - k + \beta_j(\mu_1 - \mu)} (l^{\beta_j}\omega_{j,n})^{\mu-1},\ j = 1,\dots,m$$

(the stars mean that the pair μ_1 and ν_1 is omitted). We suppose, as in [5], that there exists only a pair μ_1 and ν_1 such that $1 \le \mu_1 \le m,\ 0 \le \nu_1 \le n$ and

$$\max_{\substack{0 \le \mu \le m \\ 0 \le \nu \le n}} (\nu + \beta_j(\mu - 1)) = \nu_1 + \beta_j(\mu_1 - 1).$$

Let us denote by

$$\{v_j(t)\} := W_{j,\mu_1,n} - W_{j,\mu_1},$$

then we have

$$\{v_j(t)\} = \sum_{r=1}^{\mu-1} \binom{\mu-1}{r} c_{0,j}^{\mu-1-r} \Big(\sum_{i=n+1}^{\infty} c_{i,j} l^{i\alpha_j}\Big) \cdot$$

$$\cdot \left(\sum_{k=0}^{r-1} \Big(\sum_{i=1}^{\infty} c_{i,j} l^{i\alpha_j}\Big)^k \Big(\sum_{i=1}^{n} c_{i,j} l^{i\alpha_j}\Big)^{r-1-k} \right).$$

From relations

$$\int_0^T \Big|\sum_{i=1}^{\infty} c_{i,j} \frac{t^{i\alpha_j - 1}}{\Gamma(i\alpha_j)}\Big| dt \le V_{1,j}(T)$$

$$\int_0^T \Big|\sum_{i=r}^{\infty} c_{i,j} \frac{t^{i\alpha_j - 1 - \beta_j}}{\Gamma(i\alpha_j - \beta_j)}\Big| dt \le V_{2,j}(T),$$

it follows that

$$\int_0^T |v_j(t)| dt \le M \rho_j^{n+1} \sum_{r=1}^{\mu-1} \binom{\mu-1}{r} |c_0|^{\mu-1-r} \cdot r \frac{T^{(n+1)\alpha_j}}{(\Gamma\frac{(n+1)\alpha_j}{2} + 1)^2} V_{1,j}^r(T)$$

(22)

$$\le: \tilde{V}_j(T) \frac{1}{\Gamma(\frac{n+1}{2}\alpha_j + 1)}.$$

Denoting by

$$b_{\varepsilon,j}(t) = L_{j,n} - L_j$$

and using relations (22), we get

$$\int_0^T |b_{\varepsilon,j}(t)|dt \leq \sum_{\mu=1}^{m}{}^* \sum_{k=0}^{n}{}^* \frac{\mu|\alpha_{\mu,k}|}{\mu_1|\alpha_{\mu_1,\nu_1}|} \frac{T^{\nu_1-k+\beta j(\mu_1-\mu)}}{\Gamma(\nu_1-k+\beta_j(\mu_1-\mu)+1)}\cdot$$

$$(23)\qquad \cdot M\rho^{n+1}\sum_{r=1}^{\mu-1}\begin{pmatrix}\mu-1\\ r\end{pmatrix}|c_{0,j}|^{\mu-1-r}\frac{T^{(n+1)\alpha_j}}{\Gamma(\frac{(n+1)\alpha_j}{2}+1)^2}V_{1,j}^r(T)$$

$$\leq \frac{1}{\Gamma(\frac{(n+1)\alpha_j}{2}+1)}\tilde{b}_j(T).$$

From relations (21),(22) and (23) it follows that

$$||f(x)(A_{j,n}-A_j)||_T \leq \tilde{F}(t)\cdot\sum_{k=1}^{\infty}\frac{1}{|c_{\mu_1}|^k}(\tilde{V}_j(T)+b_j(T))\cdot$$

$$(24)\qquad \cdot\sum_{i=0}^{k-1}u_j^i(T)(u_{j,n}(T))^{k-1-i}\frac{1}{\Gamma(\frac{(n+1)\alpha_j}{2}+1)}$$

$$\leq \tilde{\tilde{F}}_{j,\varepsilon}(T)\frac{1}{\Gamma(\frac{(n+1)\alpha_j}{2}+1)},$$

where

$$\tilde{F}(t) = \max_{0\leq x\leq\lambda}\int_0^T |f_3(x,t)|dt$$

and

$$f_3(k,t) = f(x)\frac{l^{\nu_1+\beta(\mu_1-1)}}{\mu_1|\alpha_{\mu_1,\nu_1}||c_{\mu_1}|}.$$

Also,

$$u_{j,n}(T) \equiv \int_0^T |W_{j,\mu_1,n}(t)+L_{j,n}(t)|dt,$$

where

$$\{W_{j,\mu_1,n}(t)\} = W_{j,\mu_1,n};\ \ \{L_{j,n}(t)\} = L_{j,n}$$

and

$$u_j(T) \equiv \int_0^T |W_{j,\mu_1}(t)+L_j(t)|dt,$$

where

$$\{W_{j,\mu_1}(t)\} = W_{j,\mu_1};\ \ \{L_j(t)\} = L_j.$$

In the relation

$$\{R_j(\lambda - x, t)\} := g_1 \exp((\lambda - x), \omega_j) = \prod_{i=0}^{k} \exp((\lambda - x) l^{i\lambda_j - \beta_j}).$$

$$\cdot k \left(\sum_{i=0}^{\infty} (-k)^i l^{i+1} \right) \cdot \left(\sum_{i=0}^{\infty} \frac{(\lambda - x)^r}{r!} \left(\sum_{i=k+1}^{\infty} c_{i,j} l^{i\alpha_j - \beta_j} \right)^r \right); \quad j = 1, \ldots, m$$

the number k satisfies the relation $(k+1)\alpha_j - \beta_j > 0$. (If $\beta_j < 0$ then $k = -1$ it is taken $\prod_0^{-1}(\cdot) = I$ as usual). Let us suppose that α_j and β_j (for $j = 1, 2, \ldots, m$) are such that

$$\prod_{i=0}^{k} \exp((\lambda - x) l^{i\alpha_j - \beta_j}) = I + \bar{R}_j(\lambda - x, t)$$

and $\bar{R}_j(\lambda - x, t)$ is the continuous function for every j. Therefore, there exist

$$\max_{0 \leq k \leq \lambda} \int_0^T |\bar{R}_j(\lambda - x, t)| dt = R_j^M(\lambda, T).$$

So, we obtain

$$(25) \quad \int_0^T |J_{g_1}^2(\lambda, t)| dt \leq (1 + R_j^M(\lambda, T) k (2 - e^{-kT})) e^{\lambda V_{2,j}(T)} \equiv S_{g_1}(\lambda, t)$$

From the relations (18),(24) and (25) it follows that

$$(26) \qquad \int_0^T |J_{g_1}^2(\lambda, t)| dt \leq \lambda \sum_{j=1}^{m} \tilde{\bar{F}}_{j,\varepsilon}(T) S_{g_1}(\lambda, T) \frac{1}{\Gamma(\frac{(n+1)\alpha_j}{2} + 1)}.$$

So, we have proved

Lemma 1. *If the operators $y(\lambda)$ and $y_n(\lambda)$ are given by (6) then it holds that*

$$(27) \qquad B_{T,1/k}(y_n(\lambda) - y(\lambda)) \leq \lambda \sum_{j=1}^{m} \frac{1}{\Gamma(\frac{(n+1)\alpha_j}{2} + 1)} \cdot \tilde{\bar{F}}_{j,\varepsilon}(T) \times$$

$$\times S_{g_1}(\lambda, t) + \tilde{A}_j(\lambda, t) K_{g_1}^M(\lambda, T, \alpha_j, \beta_j) \frac{1}{\Gamma(\frac{(n+1)\alpha_j - \beta_j - 1}{2} + 1)}$$

where $\tilde{\bar{F}}_{j,\varepsilon}(T)$, $S_{g_1}(\lambda, t)$, $\tilde{A}_j(\lambda, t)$ and $K_{g_1}^M(\lambda, T, \alpha_j, \beta_j)$ are given by relations (24),(25),(19) and (18), respectively.

Using relation (14) and Lemma 1 it is easy to prove

Theorem 1. *If operators $x_{p,n}(\lambda)$ and $x_p(\lambda)$ are given by (10) and (7), then the measure of approximation according to A is*

$$A(x_{p,n}(\lambda) - x_p(\lambda)) \leq \sum_{j=1}^{m} \frac{1}{\Gamma(\frac{(n+1)\alpha_j - 2}{2} + 1)} \tilde{Q}^p_{g_1}(\lambda, \alpha_j, \beta_j) \equiv \delta, \tag{28}$$

where

$$\tilde{Q}^p_{g_1}(\lambda, \alpha_j, \beta_j) \geq \sum_{i=1}^{\infty} \frac{\tilde{\tilde{F}}_{j,\varepsilon}(i) S_{g_1}(\lambda, i) + \tilde{A}_j(\lambda, i) K^M_{g_1}(\lambda, i, \alpha_j, \beta_j)}{e^{ie^{i^2}}}. \tag{28'}$$

Corollary 1. *The sequence $\{x_{p,n}(\lambda)\}$ where $x_{p,n}(\lambda)$ are the approximate particular solutions of equation (1) with conditions (2) (given by (10)) converges type I' to the exact particular solution (given by (7)) in $\mathcal{F}_0$.*

4. Estimation in L

Let us suppose that the exact (and the approximate) solution of equation (1) represent functions from L and denote by

$$I(\lambda, T) = ||J(\lambda, t)||_T,$$

where

$$\{J(\lambda, t)\} = \int_0^\lambda f(x) \sum_{j=1}^{m} (A_{j,n} \exp(\lambda - x)\omega_{j,n} - A_j \exp(\lambda - x)\omega_j) dx.$$

Using estimations from the previous chapter we can order $I_j(\lambda, T)$ such that the following inequality is satisfied

$$||I(\lambda, t)||_T \leq \sum_{j=1}^{m} \frac{1}{\Gamma(\frac{(n+1)\alpha_j - 2}{2} + 1)} I_j(\lambda, T).$$

So we have

Theorem 2. *If operators $x_{p,n}(\lambda)$ and $x_p(\lambda)$ are given by (10) and (7), represents the functions from L, then the measure of approximation according to F is*

$$F(x_{p,n}(\lambda) - x_p(\lambda)) \leq \sum_{j=1}^{m} \frac{1}{\Gamma(\frac{(n+1)\alpha_j - 2}{2} + 1)} \tilde{Q}(\lambda, \alpha_j, \beta_j) \equiv \delta_L \tag{29}$$

where

$$\tilde{Q}(\lambda, \alpha_j, \beta_j) \geq \sum_{i=1}^{\infty} \frac{I_j(\lambda, i)}{e^{ie^{i^2}}}.$$

Corollary 2. *The sequence $\{x_{p,n}(\lambda)\}$ given by (10) and representing the function from L, converges in L to the operator $x_p(\lambda)$ given by (7).*

It can be remarked that the errors of approximation in $\mathcal{F}_0$ and in L, given by relations (28) and (29), respectively, are independent of the length of the interval $[0, T]$ (on t-axis). They hold for every $T > 0$.

Theorem 2 with its Corollary 2 is very important because it could be that the sequence converges (type I') in $\mathcal{F}_0$ but not in L. Also, some sequences may converge in L but not in the space of continuous functions C. Therefore, we need the following part.

5. Estimation in C

If the exact (and the approximate) solution belongs to space C, then the error of approximation can be obtained as the estimation of the difference of $|x_{p,n}(\lambda) - x_{p,n}(\lambda)|$. This was done in paper [5] where it depends of the length of the interval $[0, T]$. For small values of T it is rather good but with increasing T the error of approximation grows very fast.

In order to construct a new measure of approximation let us introduce

Definition 3. *Function $\tilde{h}$ from C is the approximation of function h, according to the functional*

$$G(h) = \sum_{i=1}^{\infty} \frac{1}{e^{ie^{i^2}}} \cdot \frac{|h|_i}{1 + |h|_i}, \quad |h|_i = \max_{0 \leq t \leq i} |h(t)|$$

with the measure of approximation δ_c, if $G(h - \tilde{h}) < \delta_c$.

Analogously as in the previous section the following Theorem can be proved.

Theorem 3. *If operators $x_{p,n}(\lambda)$ and $x_p(\lambda)$ are given by (10) and (7), represent functions from C, then the measure of approximation is of the form*

$$\delta_C = \sum_{j=1}^{m} \frac{1}{\Gamma(\frac{(n+1)\alpha_j - 2}{2})} Q_C(\lambda, \alpha_j, \beta_j),$$

where

$$Q_C(\lambda, \alpha_j, \beta_j) \geq \sum_{i=1}^{\infty} \frac{I_{j,C}(\lambda, I)}{e^{ie^{i^2}}}$$

and

$$I_{j,C}(\lambda, i) = ||J_C(\lambda, t)||_T,$$

where

$$\{J_C(\lambda, t)\} = \int_0^\lambda f(x) \sum_{j=1}^{m} (A_{j,n} \exp(\lambda - x)\omega_{j,n} - A_j \exp(\lambda - x)\omega_j) dx$$

Corollary 3. *The sequence $\{x_{p,n}(\lambda)\}$ given by (10), representing the functions from C, converges in C to the operator $x_p(\lambda)$ given by (7).*

Example. Let us observe the partial differential equation

$$\frac{\partial^4 x(\lambda, t)}{\partial \lambda^2 \partial t^2} - 2\frac{\partial^3 x(\lambda, t)}{\partial \lambda^2 \partial t} + \frac{\partial^2 x(\lambda, t)}{\partial \lambda^2} - x(\lambda, t) = 4e^\lambda \tag{30}$$

with conditions

$$\frac{\partial^2 x(\lambda, t)}{\partial \lambda^2} = e^\lambda; \quad \frac{\partial^3 x(\lambda, t)}{\partial \lambda^2 \partial t} - 2\frac{\partial^2 x(\lambda, t)}{\partial \lambda^2} = \lambda, \ \lambda > 0 \tag{31}$$

$$x(0, t) = 1 + 2t; \quad \frac{\partial x(0, t)}{\partial \lambda} = 2t^2, \ \ t > 0.$$

In the field of Mikusinski operators, the equation

$$(s - 1)^2 x''(\lambda) - x(\lambda) = (s - 4l)e^\lambda + \lambda \tag{32}$$

with conditions

$$x(0) = l + 2l^2; \quad x'(0) = 2l^2 \tag{33}$$

corresponds to equation (30). The characteristic equation of equation (32) is

$$P(\omega) \equiv (s-1)^2\omega^2 - I = 0,$$

with solutions

$$\omega_1 = \sum_{i=0}^{\infty} l^{i+1},\ \omega_2 = -\sum_{i=0}^{\infty} l^{i+1};$$

and with the approximate solutions

$$\omega_{1,n} = \sum_{i=0}^{n} l^{i+1}, \omega_{2,n} = -\sum_{i=0}^{n} l^{i+1}.$$

The coefficients A_1 and A_2 are given of the form

$$A_1 = \frac{l}{2}\sum_{k=0}^{\infty}\left(-2l + l^2 + (s-2+l)\cdot\sum_{i=0}^{\infty} l^{i+2}\right)^k \cdot (-1)^k =$$

$$= \frac{l}{2}\sum_{k=0}^{\infty} l^k(-1)^k \sum_{p=0}^{k}\binom{k}{p}(-1)^{k-p}\left(-l + l^2 + (1-2l+l^2)\sum_{i=1}^{\infty} l^{i+1}\right)^p$$

$$A_2 = -\frac{l}{2}\sum_{k=0}^{\infty} l^k(-1)^k \sum_{p=0}^{k}\binom{k}{p}(-1)^{k-p}\left(-l + l^2 + (1-2l+l^2)\sum_{i=1}^{\infty} l^{i}\right)^p.$$

The solution of equation (33) is ($c_1 = \frac{3}{2}l;\ c_2 = 2l^2 - \frac{l}{2}$)

$$x(\lambda) = \frac{3l}{2}\exp(\lambda\sum_{i=0}^{\infty} l^{i+1}) + (2l^2 - \frac{l}{2}\exp(-\lambda\sum_{i=0}^{\infty} l^{i+1}) +$$

$$+A_1\int_0^{\lambda}((s-4l)e^x + x)\exp((\lambda - x)\sum_{i=0}^{\infty} l^{i+1})dx +$$

$$+A_2\int_0^{\lambda}((s-4l)e^x + x)\exp(-(\lambda - x)\sum_{i=0}^{\infty} l^{i+1})dx$$

the approximate solution is

$$x_n(\lambda) = \frac{3l}{2}\exp(\lambda\sum_{i=0}^{n} l^{i+1}) + (2l^2 - \frac{l}{2}\exp(-\lambda\sum_{i=0}^{n} l^{i+1}) +$$

$$+A_{1,n}\int_0^{\lambda}((s-4l)e^x+x)\exp((\lambda-x)\sum_{i=0}^{n}l^{i+1})dx+$$

$$+A_{2,n}\int_0^{\lambda}((s-4l)e^x+x)\exp(-(\lambda-x)\sum_{i=0}^{n}l^{i+1})dx,$$

where

$$A_{1,n}=\frac{l}{2}\sum_{k=0}^{\infty}l^k(-1)^k\sum_{p=0}^{k}\binom{k}{p}(-1)^{k-p}\left(-l+l^2+(1-2l+l^2)\sum_{i=1}^{n}l^{i+1}\right)^p$$

$$A_{2,n}=-\frac{l}{2}\sum_{k=0}^{\infty}l^k(-1)^k\sum_{p=0}^{k}\binom{k}{p}(-1)^{k-p}\left(-l+l^2+(1-2l+l^2)\sum_{i=1}^{n}l^{i+1}\right)^p$$

The measure of approximation is

$$\delta=2\frac{1}{\Gamma(\frac{n-1}{2}+1)}\left(Q_{g_1}(\lambda,1,-1)+\tilde{Q}^p_{g_1}(\lambda,1,-1)\right),$$

where, using [4], we obtain

$$Q_{g_1}(\lambda,\alpha_j,\beta_j)\geq\sum_{i=1}^{\infty}\frac{8e^{2e^{2i}}}{e^{ie^{i^2}}}$$

and (28′), it follows that

$$\tilde{Q}^p_{g_1}(\lambda,1,-1)\geq\sum_{i=1}^{\infty}\frac{(5e+1)e^{2e^i}}{e^{ie^{i^2}}}.$$

So, we have

$$\delta\geq\frac{2}{\Gamma(\frac{n-1}{2}+1)}(5e+9)\left(\frac{e^{2e^2}}{e^e}+\frac{e}{e-1}\right).$$

The following table illustrates the measure of approximation in the space $\mathcal{F}_l$.

n	δ
11	0.195587381721E+00
12	0.162989478558E-01
13	0.125376519821E-02
14	0.895546545507E-04
15	0.597031021243E-05
16	0.373144388277E-06
17	0.219496705256E-07
18	0.121942611564E-08
19	0.641803207801E-10
20	0.320901595227E-11
21	0.152810286668E-12
22	0.694592215979E-14
23	0.301996627152E-15
24	0.125831930737E-16
25	0.503327718812E-18
26	0.193587582916E-19
27	0.716991057186E-21
28	0.256068241471E-22
29	0.882993902104E-24
30	0.294331306864E-25
31	0.949455825488E-27
32	0.296704945465E-28

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REZIME

OCENA GREŠKE ZA PARTIKULARNO REŠENJE

U radu je konstruisana ocena greške za približno rešenje nehomogene diferencijalne jednačine koje je konstruisano u radu [5]. Ocena greške u prostorima $\mathcal{F}_l$, $\mathcal{L}$ i C ne zavisi od dužine intervala $[0,T]$.

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