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THE GENERALIZED ASYMPTOTIC EXPANSION AND THE QUASIASYMPTOTIC EXPANSION AT INFINITY IN THE SPECIAL SUBSPACES OF TEMPERED DISTRIBUTIONS

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Abstract

The purpose of this paper is to compare, in the special subspaces $\mathcal{A}'\{x^{\alpha_n}\}$ of the space of tempered distributions, the notions of the generalized asymptotic expansion introduced by R. Estrado and R. P. Kanwal [1], [2], [3], and the quasiasymptotic expansion of the first and second type at ∞ introduced by S. Pilipović [4].

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1. Notions and basic results

Denote by $\mathcal{S}$ the space of rapidly decreasing smooth functions defined on the real line $\mathbf{R}$, supplied with usual topology. Its dual is the space of tempered distributions $\mathcal{S}'$ and $\mathcal{S}'_+$ is its subspace with elements supported by $[0,\infty)$.

Denote by $c_m(\lambda), m \in \mathbf{N}$, a sequence of countinuous positive functions defined on $(a_m,\infty), a_m > 0$, such that $c_{m+1}(\lambda) = o(c_m(\lambda)), \lambda \to \infty, (m \in \mathbf{N})$

and by $u_m, m \in \mathbf{N}$, a sequence from $\mathcal{S}'_+$ such that $u_m \neq 0, m = 1, ..., p, p < \infty, u_m = 0, m > p$ or $u_m \neq 0, m \in \mathbf{N}$. Denote by Λ the set of pairs of sequences $(c_m(\lambda), u_m)$.

Let $(c_m(\lambda), u_m) \in \Lambda$ and

$$\lim_{\lambda\to\infty} < \frac{u_m(\lambda x)}{c_m(\lambda)}, \varphi(x) >=< g_m(x), \varphi(x) >, \varphi \in \mathcal{S}, \tag{1}$$

where $g_m(x) \neq 0$ if $u_m \neq 0, m \in \mathbf{N}$. In this case we write $u_m \overset{q}{\sim} g_m$ at $\overset{+}{-}\infty$ with respect to $c_m(\lambda)$.

It is proved by Yu. N. Drožžinov and B. I. Zavialov [5], that in this case $g_m = C f_{\alpha_m+1}, C \neq 0$ and $c_m(\lambda) = \lambda^{\alpha_m} L_m(\lambda), \lambda > \lambda_{0,m}$, where

$$f_{p+1} = \begin{cases} \frac{H(x)x^p}{\Gamma(p+1)}, \ p > -1, \\ \qquad\qquad x \in \mathbf{R} \\ f^{(\alpha)}_{p+k+1}(x), \ p \leq -1, \ p+k > -1, \ k \in \mathbf{N}. \end{cases}$$

H is the Heaviside function and L_m is Karamata's slowly varying function, i. e. it is positive measurable function such that

$$\lim_{\lambda\to\infty} \frac{L_m(\lambda x)}{L_m(\lambda)} = 1$$

uniformly for $x \in [a, b] \subset (0, \infty)$.

Denote by Λ_1 a subset of Λ such that $(c_m(\lambda), u_m) \in \Lambda_1$ if (1) holds for all the m for which $u_m \neq 0$ and $g_m \neq 0 (m = 0, ..., p < \infty$ or $m \in \mathbf{N})$.

Let $f \in \mathcal{S}'_+$ and $(c_m(\lambda), u_m) \in \Lambda_1$ such that :

$$\lim_{\lambda\to\infty} < \frac{(f(\cdot) - \sum_{i=1}^m u_i(\cdot))(\lambda x)}{c_m(\lambda)}, \ \varphi(x) >= 0, \ \varphi \in \mathcal{S}, \tag{2}$$

for $m = 1, ..., p < \infty$ or $m \in \mathbf{N}$. Then it is said that f has the quasiasymptotic expansion at ∞ of the first type with respect to $(c_m(\lambda), u_m)$ and we write

$$f(x) \overset{q.e.}{\sim} \sum_{i=1}^{p(\infty)} u_i(x) \quad (c_i(\lambda)), \text{ at } \infty.$$

Let $f \in \mathcal{S}'_+, (c_m(\lambda), u_m) \in \Lambda_1$ with $u_m \in \mathcal{S}'_+ (m = 1, ..., p < \infty$ or $m \in \mathbf{N})$ and

$$\lim_{\lambda\to\infty} < \frac{f(\lambda x) - \sum_{i=1}^m u_i(x) c_i(\lambda)}{c_m(\lambda)}, \ \varphi(x) >= 0, \ \varphi \in \mathcal{S}, \tag{3}$$

for $m = 1, ..., p < \infty$ or $m \in \mathbf{N}$. Then it is said that f has the quasiasymptotic expansion at ∞ of the second type with respect to $(c_m(\lambda), u_m)$ and we write

$$f(\lambda x) \overset{q.e.}{\overline{\overline{\sim}}} \sum_{i=1}^{p(\infty)} u_i(x) c_i(\lambda) \quad (c_i(\lambda)) \text{ at } \infty.$$

R. Estrada and R. P. Kanwal [1] introduced several spaces of distributions that decay very fast at infinity which are subspaces of the spaces of tempered distributions. The space $\mathcal{E}(U)$ is the space of all smooth functions in U (U is an open subset of $\mathbf{R}$); convergence in $\mathcal{E}(U)$ means uniform convergence of all derivatives on compact sets. The dual space is $\mathcal{E}'(U)$, the space of distributions with compact support. The spaces $\mathcal{O}_\gamma(\mathbf{R}), \gamma \in \mathbf{R}$, consists of those smooth functions $\Phi(x)$ that satisfy $D^k\Phi(x) = O(|x|^\gamma)$ as $x \to \infty$ for every $k \in \mathbf{N}$. The space $\mathcal{O}_C(\mathbf{R})$ is the inductive limit $\lim \mathcal{O}_\gamma(\mathbf{R})$ as $\gamma \to \infty$. The elements of $\mathcal{O'}_c(\mathbf{R})$ are generalized functions that decay very fast at infinity in the distributional sense , but not necesarily in the ordinary sense.Another useful space is the space $\mathcal{K}(\mathbf{R})$ formed by those smooth functions that satisfy $D^k\Phi(x) = O(x^{r-k})$ for some r.

Another class of spaces of generalized function [2] is following.

Let $\{\Phi_n(x)\}$ be a sequence of functions defined in $(0, b]$ that satisfy (i) $|\Phi_n(x)| > 0, \quad x > 0$ (ii) $\Phi_{n+1}(x) = o(\Phi_n(x))$as $\quad x \to 0$. Let $\mathbf{C} = \mathbf{C}((0,b], \Phi_n(x))$ be the space of continuous function φ defined on $(0, b]$ such that

$$\varphi(x) = a_0\Phi_0(x) + ... + a_m\Phi_m(x) + o(\Phi_m(x)), \quad x \to 0$$

for every m. $\mathbf{C}$ is Frechet topological vector space with seminorms

$$||\varphi||_m = \sup\{|(\varphi(x) - \sum_{j=0}^{m-1} a_j\Phi_j(x))\Phi_m^{-1}(x)| : 0 < x \leq b\}, \text{for } \ m = 0, 1, 2, ...$$

Dual space is $\mathbf{C}'$ and functional $\delta_m(x) = \Phi_m^{-1}(x)\delta(x)$ given by $< \delta_m(x), \varphi(x) > = a_m$ belong to $\mathbf{C}'$. Suppose $\{\Phi_m^{(k)}\}$ is also an asymptotic sequence as $x \to 0$ for $0 \leq k \leq p$. A function $\varphi \in \mathcal{E}_p\{\Phi_n\}$ if it is of class $\mathbf{C}^p$ in $(0, \infty)$ and as $x \to 0$

$$\varphi^{(k)}(x) = a_0\Phi_0^{(k)}(x) + ... + a_m\Phi_m^{(k)}(x) + o(\Phi_m^{(k)}(x))$$

for every m and for $0 \leq k \leq p$. When $p = \infty$ we use the notation $\mathcal{E}\{\Phi_n\}$ since in the case $\Phi_n(x) = x^n$ the space $\mathcal{E}_\infty\{x^n\}$ reduced to $\mathcal{E}[0, \infty)$. As it clear , the same ideas we can use to obtain the spaces $\mathcal{K}\{\Phi_n\}$ or $\mathcal{O}_C\{\Phi_n\}$.

Theorem 1. *([3]) Let $\mathcal{A}\{x^{\alpha_n}\}$ where $\mathcal{A}$ is any of the spaces $\mathcal{E}, \mathcal{K}$ or $\mathcal{O}_C$ where* $\mathbf{Re}\alpha_n \uparrow \infty$. *Then if $f \in \mathcal{A}'\{x^{\alpha_n}\}$, it satisfies the generalized asymptotic expansion*

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{\mu(\alpha_n)\delta_n(x)}{\lambda^{\alpha_n+1}}, \quad \lambda \to \infty$$

where $\mu(\alpha_n)$ are the generalized moments

$$\mu(\alpha_n) = < f(x), x^{\alpha_n} > .$$

2. The connection between the generalized asymptotic expansion and the quasiasymptotic expansion at ∞

Let us prove the following theorem.

Theorem 2. *If $f \in \mathcal{A}'\{x^{\alpha_n}\}$ then f has the quasiasymptotic expansion of the first and the second type at ∞ and they equal.*

Proof. If $f \in \mathcal{A}'\{x^{\alpha_n}\}$ then according to Theorem 1 it has the following generalized asymptotical expansion

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{\mu(\alpha_n)\delta_n(x)}{\lambda^{\alpha_n+1}}, \quad \lambda \to \infty.$$

First let us prove that $\delta_i, i \in \mathbf{N}_0$, has quasiasymptotic behaviour at ∞ with respect to $c_i(\lambda) = \lambda^{-\alpha_i-1}$. Namely, for every $\varphi \in \mathcal{A}\{x^{\alpha_n}\}$ since $\mathcal{S} \subset \mathcal{A}\{x^{\alpha_n}\}$ it follows

$$\lim_{\lambda\to\infty} \lambda^{\alpha_i+1} < \delta_i(\lambda x), \varphi(x) >= \lim_{\lambda\to\infty} \lambda^{\alpha_i} < \delta_i(x), \varphi(x/\lambda) =$$

$$\lim_{\lambda\to\infty} \lambda^{\alpha_i} \frac{a_i}{\lambda^{\alpha_i}} = a_i, i \in \mathbf{N}_0.$$

From this it follows

$$\lim_{\lambda\to\infty} < \frac{(f\lambda x) - \sum_{i=0}^{m} \mu(\alpha_i)\delta_i(\lambda x)}{\lambda^{-\alpha_m-1}}, \varphi(x) >=$$

$$\lim_{\lambda\to\infty} < \frac{\sum_{i=0}^{\infty} \frac{\mu(\alpha_i)\delta_i(x)}{\lambda^{\alpha_i+1}} - \sum_{i=0}^{m} \mu(\alpha_i)\delta_i(\lambda x)}{\lambda^{-\alpha_m-1}}, \varphi(x) >$$

$$= \lim_{\lambda\to\infty} \lambda^{\alpha_m+1}[< \sum_{i=0}^{\infty} \frac{\mu(\lambda_i)\delta_i(x)}{\lambda^{\alpha_i+1}}, \varphi(x) > - \sum_{i=0}^{m} \frac{\mu(\alpha_i)}{\lambda^{\alpha_i+1}} a_i] =$$

$$\lim_{\lambda\to\infty} \lambda^{\alpha_m+1}[\sum_{i=0}^{m} \frac{\mu(\alpha_i)}{\lambda^{\alpha_i+1}} a_i + O(\frac{1}{\lambda^{\alpha_{m+1}+1}}) - \sum_{i=0}^{m} \frac{\mu(\alpha_i)}{\lambda^{\alpha_i+1}} a_i] =$$

$$\lim_{\lambda\to\infty} O(\frac{1}{\lambda^{\alpha_{m+1}-\alpha_m}}) = 0.$$

With $u_i(x) = \mu(\alpha_i)\delta_i(x)$ and $c_i(\lambda) = \lambda^{-\alpha_i-1}, i \in \mathbf{N}_0$, the assertion follows and the proof is complete.

Some obvious properties of the quasiasymptotic expansion at ∞ , using Theorem 2 , are given in the next theorem.

Theorem 3. *Let* $f \in \mathcal{A}'\{x^{\alpha_n}\}$. *Then*

(*i*)

$$f^{(m)}(x) \overset{q.e.}{\sim} \sum_{n=0}^{\infty} \mu(\alpha_n)\delta_n^{(m)}(x) \quad (1/\lambda)^{\alpha_n+1-m} \quad \textit{at } \infty \text{ (} \textit{in } \mathcal{A}'\{x^{\alpha_n}\}) \;\; m \in \mathbf{N},$$

where

$$\delta_n^{(m)}(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{\Gamma(\alpha_n + m - k)}{\Gamma(\alpha_n)} x^{-\alpha_n-m+k}\delta^{(k)}(x).$$

(*ii*)

$$x^m f^{(m)}(x) \overset{q.e.}{\sim} \sum_{n=0}^{\infty} \mu(\alpha_n) x^m \delta_n(x) \;\; (1/\lambda)^{\alpha_n+1+m} \quad \textit{at } \infty(\textit{in} \mathcal{A}'\{x^{\alpha_n}\}),$$

$m \in \mathbf{N}, \alpha_n \notin -\mathbf{N}$.

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REZIME

UOPŠTENI ASIMPTOTSKI RAZVOJ I KVAZIASIMPTOTSKI RAZVOJ U POSEBNIM POTPROSTORIMA TEMPERIRANIH DISTRIBUCIJA

Polazeći od potprostora $\mathcal{A}'\{x^{\alpha_n}\}$ prostora temperiranih distribucija u radu je dokazana teorema: Ako $f \in \mathcal{A}'\{x^{\alpha_n}\}$ tada f ima kvaziasimptotski razvoj prvog i drugog tipa u beskonačnosti i oni su jednaki.

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