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ON PC-PARALLEL AND PC-RECURRENT PRODUCT SPACES

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Abstract

We consider a product space with parallel or recurrent PC-tensor. Using separated coordinate system, we get some more information about its components. If, in addition, the product space is Ricci-recurrent, then, following the results of Adati and Miyazawa and Roter, we can obtain a complete classification for components of such a space.

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1. Introduction

A locally product manifold is defined as a Riemannian manifold M_n (metrics of such a manifold might not be strictly positive definite, so the manifold could also be a pseudo-Riemannian one) which is endowed by a non-degenerate tensor field F of type $(1,1)$ which itself satisfies the following conditions

$$F_i^j F_j^k = \delta_i^k,\ F_{ij} \doteq F_i^s g_{sj} = F_{ji} \tag{1.1}$$

If the tensor field satisfies

$$\nabla_k F_i^j = 0$$

with respect to the Levi-Civita connection of the (pseudo)-Riemannian space, then the locally product space is, in fact, a locally decomposable space.

Locally decomposable space admits a special coordinate system called a separated coordinate system. The separated coordinate system is not an immanent property of the decomposable space (its existence is a consequence of the definition), but it enables us to see such kind of space more clear. For example, in separated coordinate system the metric tensor and the tensor F (called the structure tensor) have such a form

$$g_{ij} = \begin{bmatrix} g_{ab} & 0 \\ 0 & g_{\alpha\beta} \end{bmatrix}; \; F_{ij} = \begin{bmatrix} g_{ab} & 0 \\ 0 & -g_{\alpha\beta} \end{bmatrix}$$

and the mixed structure tensor

$$F_i^j = \begin{bmatrix} \delta_a^b & 0 \\ 0 & \delta_\alpha^\beta \end{bmatrix}$$

where $a, b = 1, \dots, p$, $\alpha, \beta = p+1, \dots, n$ (n is the dimension of the manifold) and g_{ab} depend only on x^a, $g_{\alpha\beta}$ depend only on x^α. It means that the whole space M_n is divided in a natural way into two components: V_p with metrics g_{ab} and V_{n-p} with metrics $g_{\alpha\beta}$. M_n is a product of these two spaces. In this sense, the existence of the separated coordinate system is also natural. If we first use the fact about the existence of the separated coordinate system, we would come to locally decomposable structure of the space.

In the literature, the authors very often use the term "locally product spaces" for locally decomposable spaces. In this article, we shall do the same.

2. PC-transformations

If we have a space with involutive or anti-involutive structure which is skew-symmetric and which induces an involution (or anti-involution) of orthogonal vectors in the tangent space (hyperbolic Kaehlerian and Kaehlerian spaces), there is impossible to introduce a conformal transformation which saves the structure of the space in a natural way. It is, anyway, possible to do it in the product spaces. Such a transformation has been introduced by Tachibana and, as it saves the product structure of the space, it is called a product-conformal or, shorter, PC-transformation. A PC-transformation acts on every naturally obtained fibre of our space M_n in this way:

$$\bar{g}_{ij} = \rho g_{ij} + \sigma F_{ij},\ i,j = 1,\dots,n \tag{2.2}$$

where ρ and σ are nonzero and nonnegative functions depending on all variables of the space M_n and satisfying the following conditions:

$$\rho^2 - \sigma^2 \neq 0$$

$$\rho_i = \sigma_s F_i^s,\ \sigma_i = \rho_s F_i^s,\ \rho_i = \frac{\partial \rho}{\partial x^i},$$

$$\sigma_i = \frac{\partial \sigma}{\partial x^i}$$

If we construct Levi-Civita connection for the new metrics $\bar{g}_{ij}$, we can notice that it is symmetric, although conformal connections of Kaehlerian and hyperbolic Kaehlerian spaces are nonsymmetric.

Calculating the curvature tensor of new metrics, we can find an invariant tensor of product-conformal transformation

$$PC^i{}_{jkl} = K^i{}_{jkl} \tag{2.3}$$

$$+\alpha_2[s^i{}_{jkl} - (\alpha_1 K + \beta_1 K^*)r^i{}_{jkl} - (\alpha_1 K^* + \beta_1 K)r^{*i}{}_{jkl}]$$

$$+\beta_2[s^{*i}{}_{jkl} - (\alpha_1 K + \beta_1 K^*)r^{*i}{}_{jkl} - (\alpha_1 K^* + \beta_1 K)r^i{}_{jkl}]$$

where

$$\alpha_2 = \frac{n-4}{(n-4)^2 - \phi^2},\ \beta_2 = -\frac{\phi}{(n-4)^2 - \phi^2}$$

$$\alpha_1 = \frac{n-2}{(n-2)^2 - \phi^2},\ \beta_1 = -\frac{\phi}{(n-2)^2 - \phi^2}$$

$$s^i{}_{jkl} = \delta_k^i K_{lj} + F_k^i K_{lj}^* + g_{lj} K_k^i + F_{lj} K_k^{*i}$$

$$-\delta_l^i K_{kj} - F_l^i K_{kj}^* - g_{kj} K_l^i - F_{kj} K_l^{*i}$$

$$s^{*i}{}_{jkl} = s^s{}_{jkl} F_s^i$$

$$r^i{}_{jkl} = g_{jl}\delta_k^i - g_{jk}\delta_l^i + F_{jl}F_k^i - F_{jk}F_l^i$$

$$r^{*i}{}_{jkl} = r^s{}_{jkl} F_s^i$$

$$K_{ji}^* = K_{js}F_i^s,\ K^* = K_{ji}^* g^{ji},\ \phi = F_m^m = 2p - n$$

The tensor $PC^i{}_{jkl}$ (shorter, PC-tensor) is Tachibana's product-conformal curvature tensor. In the separated coordinate system, we have

$$(a) K_{abcd} = K_{(1)abcd}$$

$$(b) K_{\alpha\beta\gamma\delta} = K_{(2)\alpha\beta\gamma\delta}$$

where $K_{(1)abcd}$ denotes a component of curvature tensor of the space V_p $K_{(2)\alpha\beta\gamma\delta}$ denotes a component of curvature tensor of the space V_{n-p} and K_{ijkl} denotes a curvature tensor component of M_n. All other components of the curvature tensor of M_n in separated coordinate system vanish identically. From this fact, we have that only those components of the Ricci tensor which are simultaneously the components of Ricci tensors of V_p and V_{n-p} can be different from zero. Taking into account all these facts and the form of the covariant structure tensor in the separated coordinate system, we have

$$PC_{abcd} = C_{abcd}; \; PC_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} \tag{2.4}$$

where C_{abcd} and $C_{\alpha\beta\gamma\delta}$ are Weyl conformal curvature tensors of natural components V_p and V_{n-p} of the product space M_n:

$$C_{ijkl} = \frac{1}{n-2}(-g_{il}K_{jk} + g_{ik}K_{jl} - K_{il}g_{jk} + K_{ik}g_{jl}) - \frac{K}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{il}g_{jk}) \tag{2.5}$$

The all other components of the PC-tensor, which are not given by the relation (2. 3), vanish identically in the separated coordinate system.

3. Ricci-recurrent product spaces

Patterson [3] proved the following theorem:

Theorem. *If a Riemannian Ricci-recurrent space V_n is locally a product of Riemannian spaces V_p and V_{n-p}, then one of them is an Einstein space with vanishing scalar curvature and the other one is Ricci-recurrent. Conversely, a product of an Einstein space with vanishing scalar curvature and a Ricci-recurrent space is itself a Ricci-recurrent space.*

We shall use this theorem in the investigation of PC-parallel and PC-recurrent Ricci-recurrent spaces.

4. PC-parallel Ricci-recurrent spaces

In this section, besides of given theorem of Patterson, we shall use a classification for conformally symmetric Rici-recurrent spaces, given by Adati and Miyazawa [1] and the metrics for the essential case of these spaces, which has been constructed by Roter [4]. Suppose, now, that the product space M_n is PC-parallel, what means

$$\nabla_s PC_{ijkl} = 0. \tag{4.6}$$

In the separated coordinate system, where only nonvanishing components of PC-tensor are those which are given by the relation (2. 3), we obtain

$$\nabla_l C_{abcd} = 0;\ \nabla_\mu C_{\alpha\beta\gamma\delta} = 0;\ l = 1, \dots, p,\ \mu = p+1, \dots, n$$

what means that both the spaces V_p and V_{n-p} are conformally symmetric.

Suppose, further,that the space M_n is Ricci-recurrent. Then, according to Patterson's theorem, one of product components is an Einstein space with vanishing scalar curvature (for example V_{n-p}) and the other one is a Ricci-recurrent space. As the space M_n is decomposable, that means

$$\nabla_c K_{ab} = \kappa_c K_{ab},\ K_{\alpha\beta} = 0.$$

Further, as the Weyl conformal curvature tensor is given by the formula (2. 4), then, because of the relation above, $C_{\alpha\beta\gamma\delta} = K_{\alpha\beta\gamma\delta}$ and the space V_{n-p} is not only conformally symmetric, but just symmetric.

According to the classification of Adati and Miyazawa and our upper consideratioo, there yields:

Theorem 1. *If a locally product space M_n consisting of components V_p and V_{n-p} is PC-parallel and Ricci-recurrent, then one of its components (V_{n-p}) is a symmetric and Ricci-flat space. For the other component, the following cases may occur:*

a) V_p is conformally flat and recurrent space

b) V_p is a symmetric Einstein space with vanishing curvature scalar

c) V_p is essentially conformally symmetric and Ricci-recurrent with an isotropic recurrence vector and it has Roter's metric of the first type

$$ds^2 = \psi(dx^1)^2 + k_{rq} dx^r dx^q + 2dx^1 dx^p \tag{4.7}$$

where

$$\psi = \frac{1}{2(n-2)} C exp(\int Q dx^1) k_{pq} x^p x^q + a_{pq} x^p x^q$$

and (k_{pq}) *is a symmetric nonsingular matrix consisting of constants,* (a_{pq}) *is a symmetric matrix consisting of constants and* $k^{pq} a_{pq} = 0$, $Q(x^1)$ *is a function of one variable and* C *is a nonvanishing constant. The dimension of the space is greater then 3. The scalar curvature of such a metrics vanishes.*

Also, the converse theorem is valid: if we construct a product of two spaces and if one of them is symmetric and Ricci-flat and the other one satisfies one of three possibilities from Theorem 1, then such a product space will be PC-parallel and Ricci-recurrent.

As the metrics of the space in the case (c) od Theorem 1. is indefinite, then we have

Corollary 1. *If a locally-product space* V_n *has positively definite metrics and if it is PC-parallel and Ricci-recurrent, then the following cases may occur:*

a) V_n *is a symmetric Ricci-flat space*

b) One component of this space is a symmetric and Ricci-flat space and the other one is conformally flat and recurrent space.

As the metrics of the space is decomposable, we can easily obtain

$$K = K_1 + K_2$$

where K denotes the scalar curvature of the space V_n and K_1 and K_2 are scalar curvatures of the spaces V_p and V_{n-p}. According to proved Theorem 1, we have

Corollary 2. *The scalar curvature of a PC-parallel Ricci-recurrent locally product space* M_n *vanishes, except of the case when one of its components is a conformally-flat recurrent space with a Walker's [6] metrics with a nonvanishing scalar curvature. In that case, the scalar curvature of* M_n *is equal to scalar curvature of that Walker's metrics.*

In most examples of Walker's metrics its scalar curvature, however, vanishes.

5. On PC-recurrent product spaces

In this section, we shall give the classification of PC-recurrent product spaces. It will be done in the same manner as Patterson has done it for Ricci-recurrent product spaces and as Walker [6] has done it for recurrent product spaces.

Let us suppose that our product space V_n is a PC-recurrent space, that is

$$\nabla_s PC_{ijkl} = \kappa_s PC_{ijkl} \tag{5.8}$$

where (κ_s) are components of a vector field. Then according to (2. 3), in separated coordinate system there hold

$$(5.9) \qquad (a)\nabla_e C_{abcd} = \kappa_e C_{abcd}$$

$$(b)\nabla_\mu C_{\alpha\beta\gamma\delta} = \kappa_\mu C_{\alpha\beta\gamma\delta}$$

$$(c)\nabla_m u C_{abcd} = \kappa_\mu C_{abcd} = 0$$

$$(d)\nabla_e C_{\alpha\beta\gamma\delta} = \kappa_e C_{\alpha\beta\gamma\delta} = 0$$

(4. 2)(c) and (4. 2)(d) hold by the matter of the fact that components C_{abcd} and $C_{\alpha\beta\gamma\delta}$ depend only on variables (x^a) and (x^α) respectively.

From (4. 2)(c) we have that

I $\kappa_\mu = 0$ or

II $C_{abcd} = 0$ holds.

If I holds, then V_{n-p} is a conformally symmetric space. Besides, there also holds (4. 2)(d). As it is a simple product, then

1) $\kappa_e = 0$ or

2) $C_{\alpha\beta\gamma\delta} = 0$

For I and 1) we have that the whole space V_n is parallel. For I and 2) we have that the space V_{n-p} is conformally-flat and the space V_p is conformally recurrent.

For II and 1) the space V_p is conformally-flat and the space V_{n-p} is conformally recurrent. For II and 2) the whole space V_n is a PC-flat space.

We have proved

Theorem 2. *If a locally product space V_n is a PC-recurrent space, then the following cases may occur*

(a) V_n is a PC-flat space

(b) V_n is a PC-parallel space

(c) one of components of V_n is a conformally flat space and the other one is a conformally recurrent space.

6. On PC-recurrent Ricci-recurrent spaces

Let us suppose that our locally product space M_n is Ricci-recurrent

$$\nabla_k K_{ij} = \kappa_k^* K_{ij} \tag{6.10}$$

and also PC-recurrent

$$\nabla_s PC_{ijkl} = \kappa_s PC_{ijkl} \tag{6.11}$$

Then, using results of Patterson's theorem and our Theorem 2 and omitting more simple cases like PC-flat and PC-parallel space, we have the following two possibilities:

I That component of locally product space which is conformally flat is an Einstein space with vanishing scalar curvature; then it is a flat (Euclidean) space; the other component is a conformally-recurrent Ricci-recurrent space.

II That component of locally-product space which is conformally-flat is Ricci-recurrent; then it is recurrent. The other one is conformaly-recurrent Einstein space with vanishing scalar curvature; then it is also a recurrent space and, according to Adati and Miyazawa [2], if its metrics is positively definite, it is a space of constant curvature.

We have just proved

Theorem 3. *If a locally product space V_n is PC-recurrent and Ricci-recurrent ((5. 1) and (5. 2)) and if at least one of recurrence vectors (κ_k^*), (κ_k) is a gradient vector field, then one of following cases occur:*

(1) M_p is an Euclidean space and for V_{n-p} holds one of following possibilities:

(a) V_{n-p} is a recurent space, if the recurrence vector equal to the recurrence vector κ_k^*

(b) $K = 0$ *and* $K_{\alpha\beta}K^{\alpha\beta} = 0$

(c) V_{n-p} *is a conformally-flat Ricci-recurrent space*

(d) κ^*_k *is an isotropic vector field and* V_{n-p} *is essentially conformally-recurrent Ricci-recurrent space. Than it possesses Roter's metrics of second type:*

(6.12) $$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = Q(dx^{p+1})^2 + k_{\gamma\delta}dx^\gamma dx^\delta + 2dx^{p+1}dx^n$$

(6.13) $$\partial_\gamma\partial_\delta Q = Ak_{\gamma\delta} + Bc_{\gamma\delta}$$

where $[k_{\gamma\delta}]$ *is a symmetric nonsingular matrix,* $[c_{\gamma\delta}]$ *is a symmetric nonzero matrix such that* $k^{\gamma\delta}c_{\gamma\delta} = 0$ *and* A *and* B *are functions which do not depend on* x^n*, so that, except of (5. 4), there hold*

(6.14) $$A \neq 0, B \neq 0,\ \partial_\alpha A \neq 0, \partial_\alpha B \neq 0\ A\partial_\alpha B - B\partial_\alpha A = 0$$

(2) M_p *is a recurrent space and* V_{n-p} *is an Einstein space with vanishing scalar curvature.*

The converse theorem is also true.

For Riemannian product spaces (those with strictly positively definite metrics) according to Theorem 3, there will hold:

Corollary 3. *If a Riemannian locally product space* M_n $(n > 3)$ *is PC-recurrent with a gradient recurrence vector and if it is also Ricci-recurrent, then one of the following cases occur:*

(1) V_p *is an Euclidean space and for* V_{n-p} *there holds one of the following possibilities*

(a) V_{n-p} *is a flat extension of* V_2 *(or* V_2 *itself) if the recurrence vector* (κ_k) *is equal to the recurrence vector* (κ^*_k)

(b) $K = 0$ *and* $K_{\alpha\beta}K^{\alpha\beta} = 0$

(2) V_p *and* V_{n-p} *are both flat extensions of* V_2.

The scalar curvature of PC-recurrent Ricci-recurrent Riemannian space, as well as scalar curvatures of its both components, vanish.

Because of decomposability of the metrics, there also holds

Corollary 4. *Scalar curvature of PC-recurrent Ricci-recurrent product space V_n is equal to scalar curvature of its component V_p.*

7. Almost Einstein PC-recurrent spaces

If a product space M_n is an Einstein space, then each its component is an Einstein space. Conversely, if we make a product of two Einstein metrices, that product may not be an Einstein space. Generally, in any coordinate system such a product will have a property

$$K_{ij} = cg_{ij} + rF_{ij} \tag{7.15}$$

Conversely, if the product space V_n satisfies (6. 1), then each of its components will be an Einstein space.

If M_n is an Einstein space, it will also satisfy (6. 1).

Definition 1. *A product space satisfying (6. 1) is called an almost Einstein space.*

Let us suppose that our product space M_n is an almost Einstein space and, moreover, that it is PC-recurrent. Them, according to the definition of an almost Einstein space and Theorem 3, we have

1) One of components of the product space M_n is an Einstein space and, moreover, it is a conformally flat space. By the last fact, the curvature tensor of this component can be expressed in therms of the Ricci tensor, the scalar curvature and metric tensor. But, as the space is an Einstein space, this reduces to the scalar curvature and metric tensor only. The scalar curvature of an Einstein space is a global constant. According to (2. 4), the space is a space of constant curvature.

2) The other component is an Einstein space and it is also a conformally recurrent space.

Using the result of Adati and Miyazawa [2] about conformally recurrent Einstein spaces, we have

Theorem 4. *If an almost Einstein product space V_n is PC-recurrent, then its component V_p is a space of constant curvature. For the other component V_{n-p}. one of the following cases occur:*

a) V_{n-p} is conformally flat;

b) V_{n-p} is conformally-recurrent with an isotropic recurrence vector

c) V_{n-p} is a space of constant curvature.

It is almost obvious that, if the case c) occurs, then M_n is also a space of constant curvature.

If a locally product space is a space of constant curvature, then both of its components are spaces of constant curvature. Conversely, if we make a product of two metrices of constant curvature, the product may not be a space of constant curvature.

Definition 2. *A product space whose components are both spaces of constant curvature is called a space of almost constant curvature.*

From Theorem 4, we can easily obtain:

Corollary 5. *If a Riemannian almost Einstein space V_n is PC-recurrent, then it is a space of almost constant curvature.*

8. Almost Ricci-recurrent PC-recurrent spaces

As we can see from the Patterson's theorem, a product of two Ricci-recurrent space cannot be a Ricci-recurrent space.

Definition 3. *A locally-product space whose both components are Ricci-recurrent spaces is called an almost Ricci-recurrent space.*

The fact of almost Ricci-recurrence can be expressed in the following way:

$$\nabla_k K_{ij} = \kappa_k K_{ij} + \lambda_k K^*_{ij} \tag{8.16}$$

It can easily be seen that a Ricci-recurrent product space is a special case of an almost Ricci-recurrent space.

Using our Theorem 2, results of Adati and Miyazawa and Roter's metrics, we obtain

Theorem 5. *If a locally product space M_n is PC-recurrent and almost Ricci-recurrent, then one of its components (V_p) is a conformally flat Ricci-recurrent space and the other one (V_{n-p}) is a conformally-recurrent Ricci-recurrent space. Then one of the following cases occurs:*

(a) V_p is conformally flat and recurrent, V_{n-p} is recurrent if the recurrence vector of its conformal curvature tensor is equal to the recurrence vector of the Ricci tensor. Then the space M_n is said to be an almost recurrent space.

(b) V_p is conformally flat and recurrent, V_{n-p} satisfies $K = 0$ and $K_{\alpha\beta}K^{\alpha\beta} = 0$.

(c) V_p is conformally flat and recurrent, V_{n-p} is conformally flat and recurrent; M_n is PC-flat and almost recurrent.

(d) V_p is conformally flat and recurrent, V_{n-p} has metrics as in (6. 3) (Roter's metrics of the second type)

(e) V_p is a symmetric Ricci-flat space, V_{n-p} is a recurrent space

(f) V_p is symmetric and Ricci-flat, V_{n-p} has vanishing scalar curvature and $K_{\alpha\beta}K^{\alpha\beta} = 0$. Then M_n has vanishing scalar curvature.

(g) V_p is symmetric and Rici-flat, V_{n-p} has Roter's metric of the second type, like in (6. 3)

(h) V_p has Roter's metrics of the first type like in (4. 2) and V_{n-p} is a recurrent space

(i) V_p has Roter's metrics of the first type and V_{n-p} has vanishing scalar curvature and $K_{\alpha\beta}K^{\alpha\beta} = 0$. Then V_n has vanishing scalar curvature.

(j) V_p has Roter's metrics of the firs type and V_{n-p} has Roter's metric of the second type.

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REZIME

O PC-PARALELNIM I PC-REKURENTNIM PRODUKT-PROSTORIMA

U radu se posmatra produkt-prostor sa paralelnim ili rekurentnim PC-tenzorom. Korišćenjem separiranog koordinatnog sistema, dobija se više informacija o geometrijskim osobinama prirodnih komponenata ovog prostora. Ako se, osim ovog, pretpostavi još i da je prostor Riči-rekurentan, prema rezulatima iz navedene literature dobija se kompletna klasifikacija komponenata ovakvog prostora.

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