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CURVATURE PINCHING FOR MINIMAL SUBMANIFOLDS IN A SPHERE

Li Haizhong [1]
Department of Mathematics, Zhengzhou University
Zhengzhou 450052, China

Abstract

By use of the modified Simons' method, a pinching theorem for minimal submanifolds in a sphere is obtained. Our theorem is the improvement of Simons' theorem and Shen Yibing's theorem.

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1. Introduction

Let M^n be an n-dimensional compact minimal submanifold in a Euclidean unit sphere S^{n+p} of dimension $n+p$. It is well known, that if the length square $||\sigma||^2$ of the second fundamental form of M^n satisfies

$$(1) \qquad ||\sigma||^2 \leq n/(2-\frac{1}{p})$$

everywhere, then either $||\sigma||^2 = 0$ (i.e. M^n is totally geodesic) or $||\sigma||^2 = n/(2-\frac{1}{p})$. In the latter case M^n is either a Clifford hypersurface or a Veronese surface in S^4 ([5, 2]). In [7], Shen Yibing modified Simons' method and improved the above Simons' pinching constant. In fact, he obtained

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Theorem A (Theorem 1.1 of [7]). *Let M^n be an n (≥ 2)-dimensional compact minimal submanifold in S^{n+p} with*

$$(2) \qquad ||\sigma||^2 \leq n/(1+\sqrt{\frac{n-1}{2n}}).$$

Then M^n is either a totally geodesic submanifold or a Veronese in S^4.

In this paper, we greatly improve the above Theorem A. In fact, we establish the following result:

Theorem 1. *Let M^n be an $n(\geq 2)$-dimensional compact minimal submanifold in S^{n+p} with*

$$(3) \qquad ||\sigma||^2 \leq \frac{n(3n-2)}{5n-4}.$$

Then M^n is either a totally geodesic submanifold or a Veronese surface in S^4.

Remark 1. Our pinching constant $\frac{n(3n-2)}{5n-4}$ is independent of the codimension p of M^n and is not less than Simons' constant $n/(2-\frac{1}{p})$ in the case $p \geq 3-2/n$ (i.e. $n=2$, $p \geq 2$, $n \geq 3$, $p \geq 3$). Our pinching constant is not less that Shen Yibing's pinching constant $n/(1+\sqrt{\frac{n-1}{2n}})$.

Remark 2. In [4], Sasaki,M. studied the same problem. By Gauss equation his condition of Theorem 1 in [4] is equivelent to $||\sigma||^2 \leq n/(2-\frac{2}{(n-1)(n+1)})$. Obviously our condition (3) is better than his.

Remark 3. When $n=2$, our Theorem 1, Theorem A and Sasaki's Theorem all have the same result. In this case, the condition of the Theorem is equivalent to the Gauss curvature $K \geq \frac{1}{3}$. This result has been proved by Benko-Kothe-Semmler-Simon ([1]) and Itoh ([3]) independently.

In [3] and [6], it was proved that if M^n is a compact minimal submanifold of S^{n+p} with a sectional curvature $> \min(\frac{p-1}{2p-1}, \frac{n}{2(n+1)})$, then M^n is totally geodesic. A further problem is as follows: Can we find a pincing constant c_s depending on the dimension n and the codimension p such that $c_s \leq \min(\frac{p-1}{2p-1}, \frac{n}{2(n+1)})$? By use of our Theorem 1, we give a partial answer to this problem.

Theorem 2. *Let M^n be a compact minimal submanifold in S^{n+p} with the constant scalar curvature. If the sectional curvature of M^n is not less that*

$\frac{1}{2} - \frac{3n-2}{2p(5n-4)}$ *everywhere, then M^n is either a totally geodesic submanifold or a Veronese surface in S^4.*

Remark 4. It is clear that the pinching constant $\frac{1}{2}-\frac{3n-2}{2p(5n-4)} \le \min(\frac{p-1}{2p-1}, \frac{n}{2(n+1)})$ if $3 - 2/n \le p \le \frac{(3n-2)(n+1)}{5n-4}$.

Remark 5. Theorem 2 has improved Proposition 4.1 of [7].

2. Preliminaries

In this paper, we shall use the same notation as in [7]. Let M^n be an n-dimensional compact minimal submanifold in a unit sphere S^{n+p} of the dimension $n+p$. We choose a local field of orthonormal frames $e_1, ..., e_{n+p}$ in S^{n+p} in such a way that, when restricted to M^n, $e_1, ..., e_n$ are tangent to M^n. Let $\omega^1, ..., \omega^{n+p}$ be their dual coframes. The following convention on the range of indices will be used

$$1 \le A, B, C, ... \le n+p; \quad 1 \le i, j, k, ... \le n;$$

$$n+1 \le \alpha, \beta, ... \le n+p.$$

The second fundamental form of M^n in S^{n+p} is

$$\sigma = \sum_{\alpha,i,j} h_{ij}^{\alpha} \omega^i \omega^j e_\alpha$$

and

$$||\sigma||^2 = \sum_{\alpha,i,j} (h_{ij}^{\alpha})^2$$

is the square of the length of σ.
Let

$$UM = \bigcup_{x \in M} UM_x \quad \text{and} \quad UM_x = \{u \in TM_x : ||u|| = 1\}.$$

Thus $UM \to M$ is the unit tangent bundle over M^n. We define a function $f : UM \to R$ by

$$f(u) = ||\sigma(u,u)||^2 = \sum_{\alpha} (\sum_{i,j} h_{ij}^{\alpha} u^i u^j)^2,$$

for $u = \sum_i u^i e_i \in UM$. Since UM is compact, f attains its maximum at the vector in UM. Suppose that this vector is $v = \sum_i v^i e_i \in UM_{x_0}$, $x_0 \in M^n$ Assume $e_i = v$ at x_0 and

(4) $$b_{ij} = \sum_\alpha h_{11}^\alpha h_{ij}^\alpha.$$

From the maximum conditions of the function f at v, we have at x_0 (see Lemma 1.1 of [7]);

(5) $$f(v) = b_{11} = \max_{u\in UM}\{||\sigma(u,u)||^2\},$$

(6) $$\sum_\alpha (h_{11i}^\alpha)^2 + \sum_\alpha h_{11}^\alpha h_{11ii}^\alpha \le 0,$$

(7) $$b_{ij} = 0 \quad (i \ne j),$$

(8) $$2\sum_\alpha (h_{1k}^\alpha)^2 + b_{kk} - f(v) \le 0 \quad (k \ne 1).$$

3. Proofs

Proof of Theorem 1. All the calculations below will be made out at the point x_0. Summing up over i in (6), by the Ricci identity and (7), we get (see (2.3) of [7]):

(9) $$0 \ge nf(v) + 2\sum_{\alpha,k\ne 1} b_{kk}(h_{1k}^\alpha)^2 -$$

$$-2f(v)\sum_{\alpha,k\ne 1}(h_{1k}^\alpha)^2 - \sum_{k\ne 1} b_{kk}^2 - f(v)\sum_\alpha (h_{11}^\alpha)^2,$$

which is the fundamental inequality used below. The key problem is how to estimate from below the second term on the right-hand side: $2\sum_{\alpha,k\ne 1} b_{kk}(h_{1k}^\alpha)^2$. We obtain a better result because we give a better lower bound.

In [7], Shen Yibing gave the following estimation $2\sum_{\alpha,k\ne 1} b_{kk}(h_{1k}^\alpha)^2$ (see (2.7) of [7]):

(10) $$2\sum_{\alpha,k\ne 1} b_{kk}(h_{1k}^\alpha)^2 \ge -\frac{1}{a}f(v)\sum_{\alpha,k\ne 1}(h_{kk}^\alpha)^2 - af(v)\sum_{\alpha,k\ne 1}(h_{1k}^\alpha)^2,$$

where $a > 0$ is an arbitrary real number.

On the other hand, by $b_{kk}^2 \leq f(v)\sum_\alpha (h_{kk}^\alpha)^2 \leq f(v)^2$, we have $(f(v)+b_{kk})(f(v)-b_{kk}) \geq 0$. Combining this with (8), we get $f(v)+b_{kk} \geq 0$, i.e. $b_{kk} \geq -f(v)$. Therefore we have the following estimation:

$$2\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^\alpha)^2 \geq -2f(v)\sum_{\alpha,k\neq 1}(h_{1k}^\alpha)^2. \tag{11}$$

Combining (10) with (11), we obtain

$$2\sum_{\alpha,k\neq 1} b_{kk}^2(h_{1k}^\alpha)^2 = b\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^\alpha)^2 + (2-b)\sum_{\alpha,k\neq 1} b_{kk}(h_{1k}^\alpha)^2$$

$$\geq -\frac{bf(v)}{2a}\sum_{\alpha,k\neq 1}(h_{kk}^\alpha)^2 - \frac{abf(v)}{2}\sum_{\alpha,k\neq 1}(h_{1k}^\alpha)^2 - (2-b)f(v)\sum_{\alpha,k\neq 1}(h_{1k}^\alpha)^2$$

$$= -\frac{bf(v)}{2a}\sum_{\alpha,k\neq 1}(h_{kk}^\alpha)^2 - (2-b+\frac{ab}{2})f(v)\sum_{\alpha,k\neq 1}(h_{1k}^\alpha)^2, \tag{12}$$

where $a>0$ and $0\leq b\leq 2$.
Substituting (12) into (9) and using $b_{kk}^2 \leq f(v)\sum_\alpha (h_{kk}^\alpha)^2$, we get

$$0 \geq nf(v) - (4-b+\frac{ab}{2})f(v)\sum_{\alpha,k\neq 1}(h_{1k}^\alpha)^2 - \tag{13}$$

$$-(\frac{b}{2a}+1)f(v)\sum_{\alpha,k\neq 1}(h_{kk}^\alpha)^2 - f(v)\sum_\alpha (h_{11}^\alpha)^2!$$

(5) implies that

$$\frac{b}{2an}\sum_{\alpha,k\neq 1}(h_{kk}^\alpha)^2 \leq \frac{(n-1)b}{2an}\sum_\alpha (h_{11}^\alpha)^2, \tag{14}$$

therefore we have from (13)

$$0 \geq nf(v) - (4-b+\frac{ab}{2})f(v)\sum_{\alpha,k\neq 1}(h_{1k}^\alpha)^2 -$$

$$-(\frac{b}{2a}+1-\frac{b}{2an})f(v)\sum_{\alpha,k\neq 1}(h_{kk}^\alpha)^2 - (1+\frac{(n-1)b}{2an})f(v)\sum_\alpha (h_{11}^\alpha)^2 =$$

$$= f(v)[n-(4-b+\frac{ab}{2})\sum_{\alpha,k\neq 1}(h_{1k}^\alpha)^2 - (1+\frac{(n-1)b}{2an})\sum_{\alpha,i}(h_{ii}^\alpha)^2]. \tag{15}$$

Let $4-b+ab/2 = 2(1-(n-1)b/2an)$, i.e. $b = \frac{4an}{(3n-2)-n(a-1)^2}$. Noting

$$||\sigma||^2 = \sum_{\alpha,i,j}(h_{ij}^{\alpha})^2 \geq \sum_{\alpha,i}(h_{ii}^{\alpha})^2 + 2\sum_{\alpha,k\neq 1}(h_{1k}^{\alpha})^2,$$

we obtain from (15)

$$0 \geq f(v)[n-(1+\frac{(n-1)b}{2an})||\sigma||^2(x_0)]$$

$$(16) \qquad = f(v)[n-(1+\frac{2(n-1)}{(3n-2)-n(a-1)^2})||\sigma||^2(x_0)].$$

Let $a=1$, then (16) becomes

$$(17) \qquad 0 \geq f(v)[n-\frac{5n-4}{3n-2}||\sigma||^2(x_0)].$$

Thus, it follows from (3) and (17) that either $f(v)=0$ or

$$(18) \qquad ||\sigma||^2(x_0) = \frac{n(3n-2)}{5n-4}.$$

If $f(v)=0$, then $||\sigma(u,u)||^2 = 0$ for any $u \in UM$, so that M^n is totally geodesic. If $f(v) \neq 0$, then (18) holds, so that (9)-(17) are all equalities in the case $a=1$ and $b=\frac{4n}{3n-2}$. Making the same discussion as in [7], we conclude that $n=2$ and Theorem 1 follows directly from Theorem B of [1]. Theorem 1 is proved completely. □

Proof of Theorem 2. Let K_M be the infimum of the sectional curvatures of M^n. It is easy to see that (cf. [6])

$$(19) \qquad 2\sum_{\alpha,i,j,k,l} h_{ij}^{\alpha}(h_{kl}^{2}R_{lijk} + h_{li}^{\alpha}R_{lkjk}) \geq 2nK_M||\sigma||^2.$$

By Lemma 1.2 in [7] (i.e. (1.19) in [7]), we have

$$(20) \qquad 0 \geq \int_{M^n} ||\sigma||^2(2nK_M + \frac{1}{p}||\sigma||^2 - n)dv.$$

Since the scalar curvature of M^n is a constant, then $||\sigma||^2 = \text{const}$. Now we assume that

$$(21) \qquad ||\sigma||^2 > \frac{n(3n-2)}{5n-4}.$$

Substituting (21) into (20) and using the condition of Theorem 2, we have

$$0 > \int_{M^n} ||\sigma||^2 (K_M - (\frac{1}{2} - \frac{3n-2}{2p(5n-4)})dv > 0. \tag{22}$$

This contradiction implies that (21) is false. Therefore, $||\sigma||^2 \leq \frac{n(3n-2)}{5n-4}$, so that Theorem 2 follows from Theorem 1. □

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REZIME

GRANIČNA KRIVINA ZA MINIMALNU PODMNOGOSTRUKOST NA SFERI

U radu je dokazana granična teorema za minimalnu podmnogostrukost na sferi korišćenjem modifikovanog Simonsovog metoda. Ova teorema je poboljšanje Simonsove teoreme i Shen Yibingove teoreme.

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