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ON REDUCIBILITY OF TOTALLY SYMMETRIC n-QUASIGROUPS

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Abstract

An n-quasigroup is completely reducible if it can be represented by $n-1$ binary quasigroups. In this paper complete reducibility of totally symmetric (TS) n-quasigroups and n-loops was considered, $n \geq 3$. It is proved that for every completely reducible TS n-quasigroup (Q, f), there exist an abelian group $(Q, +)$, a permutation φ of Q and $b \in Q$ such that for all $x_1^n \in Q$ $f(x_1^n) = \varphi^{-1}(-\sum_{i=1}^{n} \varphi x_i + b)$. It is also proved that every TS n-loop (Q, f), is an n-group with unit iff (Q, f) is completely reducible (in [3] an incorrect proof of this theorem was given). There exists a completely reducible TS n-loop of order q iff $q = 2^k$, $k \in N$. A corollary of this is that for every $q \equiv 2, 4$ $(mod\ 6)$, $q \neq 2^k$, there exists a TS 3-loop of order q which is not completely reducible.

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1. Definitions and Notations

First we give some basic definitions and notations. Other notions from the theory of n-quasigroups can be found in [1].

The sequence $x_m, x_{m+1}, \dots, x_n$ we shall denote by x_m^n or $\{x_i\}_{i=m}^n$. If $m > n$, then x_m^n will be considered empty. The sequence $x, x, \dots, x$ (m times) will be denoted by $\overset{m}{x}$. If $m \leq 0$, then $\overset{m}{x}$ will be considered empty.

An n-ary groupoid (n-groupoid) (Q, f) is called an n-quasigroup if the equation $f(a_1^{i-1}, x, a_{i+1}^n) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in \{1, \dots, n\} = N_n$.

An n-quasigroup is called an n-loop if there exists $e \in Q$ such that $f(\overset{i-1}{e}, x, \overset{n-i}{e}) = x$ for all $x \in Q$ and all $i \in N_n$, and e is called a unit of that n-loop.

An n-quasigroup (Q, f) is isotopic to an n-quasigroup (Q, g) if there exists a sequence $T = (\alpha_1^{n+1})$ of permutations of Q such that the following identity

$$g(x_1^n) = \alpha_{n+1}^{-1} f(\{\alpha_i x_i\}_{i=1}^n)$$

holds.

By S_n we denote the symmetric group of degree n.

An n-quasigroup (Q, f) such that

$$f(x_1^n) = x_{n+1} \Longleftrightarrow f(\{x_{\sigma i}\}_{i=1}^n) = x_{\sigma(n+1)}$$

for all $\sigma \in S_{n+1}$ is called totally symmetric (TS).

An n-quasigroup (Q, f) is called (i,j)-associative if the following identity holds

$$f(x_1^{i-1}, f(x_i^{i+n-1}), x_{i+n}^{2n-1}) = f(x_1^{j-1}, f(x_j^{j+n-1}), x_{j+n}^{2n-1}).$$

An n-quasigroup which is (i,j)-associative for all $i, j \in N_n$ is called associative. An associative n-quasigroup is an n-group.

2. Reducibility of TS n-quasigroups

First we shall define reducibility and complete reducibility of n-quasigroups ([2]).

Definition 1. *An n-quasigroup (Q, f), $n \geq 3$, is called reducible if there exist a p-quasigroup (Q, g) and a q-quasigroup (Q, h) such that for some*

$i \in N_n$ and all $x_1^n \in Q$

$$f(x_1^n) = g(x_1^{i-1}, h(x_i^{i+q-1}), x_{i+q}^n).$$

We shall introduce the following notation. For example, if g_1^5 are binary operations on a set Q and if

$$w = g_1(g_2(x_1^2), g_3(g_4(x_3, g_5(x_4^5)), x_6)), \tag{1}$$

then (1) we shall write as

$$w = g_1 g_2 x_1^2 g_3 g_4 x_3 g_5 x_4^6,$$

or

$$w = g_1^1 g_2^1 g_3^3 g_4^3 g_5^4 x_1^6,$$

where each of the upper indices of operations g_i denotes the index of the first free variable which comes after g_i.

Definition 2. *An n-quasigroup (Q, f), $n \geq 3$, is completely reducible if there exist binary quasigroups $(Q, g_1), \ldots, (Q, g_{n-1})$ such that for some $k_1^{n-1} \in N_{n-1}$ and all $x_1^n \in Q$*

$$f(x_1^n) = \{g_i^{k_i}\}_{i=1}^{n-1} x_1^n,$$

where k_i denotes the index of the first free variable which comes after g_i.

The numbers k_1^{n-1} from the preceding definition obviously satisfy $k_1 = 1$, $k_i \leq k_{i+1}$, $k_i \leq i$. A 3-quasigroup is reducible iff it is completely reducible.

Lemma 1. *If a TS n-quasigroup (Q, f), $n \geq 3$, is completely reducible, then there exist an abelian group $(Q, +)$, a permutation φ of Q and $a, b \in Q$ such that for all $x, y, z \in Q$*

$$f(x, y, z, \overset{n-3}{a}) = \varphi^{-1}(-\varphi x - \varphi y - \varphi z + b).$$

Proof. In the proof of Theorem 1 from [4] we have shown that for some $a \in Q$ $f(x, y, z, \overset{n-3}{a}) = h_1(h_2(x, y), z)$ (or $f(x, y, z, \overset{n-3}{a}) = h_1(x, h_2(y, z))$), where h_1, h_2 are quasigroups. We have also proved that in the first case (the other one is analogous), there exists an abelian group $(Q,+)$ such that $h_1(x,y) = R_1(x + R_2y)$, $h_2(x,y) = R_2x + R_2y$, $R_ix = h_i(x,a)$, $i = 1,2$, which implies

$$f(x,y,z,\overset{n-3}{a}) = R_1(R_2x + R_2y + R_2z).$$

But f is TS, hence

$$R_2x + R_2y + R_2z = R_1^{-1}u \iff R_2u + R_2y + R_2z = R_1^{-1}x$$

and for $u = a$

$$R_2x + R_1^{-1}x = R_1^{-1}a + R_2a.$$

Since $R_2a = h_2(a,a) = R_2a + R_2a$, we get $R_2a = 0$. So $R_1^{-1}x = -R_2x - b$, where $b = -R_1^{-1}a$. But $f(x,y,z,\overset{n-3}{a}) = u$ is equivalent to $R_2x+R_2y+R_2z = R_1^{-1}u$, hence $R_2x + R_2y + R_2z = -R_2u - b$, that is, $R_2^{-1}(-R_2x - R_2y - R_2z + b) = u$. □

For $n = 3$ Lemma 1 gives the following corollary.

Corollary 1. *For every completely reducible TS 3-quasigroup (Q,f), there exist an abelian group $(Q,+)$, a permutation φ of Q and $b \in Q$ such that for all $x_1^3 \in Q$*

$$f(x_1^3) = \varphi^{-1}(-\varphi x_1 - \varphi x_2 - \varphi x_3 + b).$$

Theorem 1. *For every completely reducible TS n-quasigroup (Q,f), there exist an abelian group $(Q,+)$, a permutation φ of Q and $b \in Q$ such that for all $x_1^n \in Q$*

$$f(x_1^n) = \varphi^{-1}(-\sum_{i=1}^{n} \varphi x_i + b).$$

Proof. We shall use induction on n to prove the theorem.

For $n = 3$ from Corollary 1 we have that for every completely reducible 3-quasigroup (Q,f) there exist an abelian group $(Q,+)$, a permutation φ of Q and $b \in Q$ such that

$$f(x_1^3) = \varphi^{-1}(-\varphi x_1 - \varphi x_2 - \varphi x_3 + b).$$

Assume that for every completely reducible TS $(n-1)$-quasigroup (Q,f) there exist an abelian group $(Q,+)$ a permutation φ of Q and $b \in Q$ such that

$$f(x_1^{n-1}) = \varphi^{-1}(-\sum_{i=1}^{n-1} \varphi x_i + b).$$

Let (Q, f) be a completely reducible TS n-quasigroup, $f(x_1^n) = \{g_i^{k_i}\}_{i=1}^{n-1} x_1^n$. Then it is not difficult to prove that the term $w = \{g_i^{k_i}\}_{i=1}^{n-1} x_1^n$ always contains a subterm of the form $g_i(x_j, x_{j+1})$. To simplify the notation, we can take, without loss of generality, that $i = 2$, $j = 1$. So we assume that f is of the form $f(x_1^n) = g_1(g_2(x_1^2), \ldots)$.

We define a new $(n-1)$-quasigroup f' by $f'(x_2^n) = f(a, x_2^n)$ where $a \in Q$. f' is obviously completely reducible TS $(n-1)$-quasigroup. Now

$$f'(L_2^{-1} g_2(x_1^2), x_3^n) = f(x_1^n),$$

where $L_2 x = g_2(a, x)$. But f' is of arity $n - 1$, hence by the induction hypothesis

$$f'(x_2^n) = \varphi^{-1}(-\sum_{i=2}^{n} \varphi x_i + b),$$

that is,

$$f(x_1^n) = \varphi^{-1}(-\varphi L_2^{-1} g_2(x_1^2) - \sum_{i=3}^{n} \varphi x_i + b). \tag{2}$$

Putting in (2) $x_i = a$, $i = 3, \ldots, n$, we get

$$f(x_1^2, \overset{n-2}{a}) = -\varphi^{-1}(-\varphi L_2^{-1} g_2(x_1^2) - (n-2)\varphi a + b),$$

and putting in (2) $x_1 = a$, $x_i = a$, $i = 4, \ldots, n$ we obtain

$$f(a, x_2^3, \overset{n-3}{a}) = \varphi^{-1}(-\varphi x_2 - \varphi x_3 - (n-3)\varphi a + b).$$

But f is TS, hence

$$\varphi^{-1}(-\varphi L_2^{-1} g_2(x, y) - (n-2)\varphi a + b) =$$

$$\varphi^{-1}(-\varphi x - \varphi y - (n-3)\varphi a + b)$$

and

$$\varphi L_2^{-1} g_2(x, y) = \varphi x + \varphi y - \varphi a.$$

So we have obtained that

$$f(x_1^n) = \varphi^{-1}(-\sum_{i=1}^{n} \varphi x_i + c)$$

where $c = \varphi a + b$. □

In [3] a theorem was given, stating that every TS n-loop (Q, f) is an n-group with unit iff (Q, f) is completely reducible, but a simple "proof" of this theorem in [3] is based on the incorrect Lemma 4 ([3], p.181). A counterexample to Lemma 4 from [3] was given in [5]. Nevertheles, the theorem is true and we shall prove it here.

Theorem 2. *Let (Q, f) be a TS n-loop (Q, f), $n \geq 3$. (Q, f) is an n-group with unit iff (Q, f) is completely reducible.*

Proof. Let (Q, f) be a TS n-loop with unit $a \in Q$ which is completely reducible.

From Theorem 1 it follows that there exist an abelian group $(Q, +)$, a permutation φ of Q and $b \in Q$ such that

$$f(x_1^n) = \varphi^{-1}(-\sum_{i=1}^{n} \varphi x_i + b).$$

Putting in the preceding equation $x_i = a$, $i = 3, \ldots, n$, we get

$$x_1 * x_2 = f(x_1^2, \overset{n-2}{a}) = \varphi^{-1}(-\varphi x_1 - \varphi x_2 + c),$$

where $c = -(n-2)a + b$. $(*)$ is obviously a binary TS loop with unit a. We see that the loop $(*)$ isotopic to the group $(+)$, which by Albert's theorem implies that they are isomorphic.

We have obtained that $(+)$ and $(*)$ are both TS groups, hence from $x + y = z \iff x + z = y$ we get that $(\forall x \in Q)\ x + x = 0$. Hence $(Q, +)$ is a boolean group of order 2^k, $k \in N$.

Since $(+)$ is boolean from

$$f(x_1^n) = \varphi^{-1}(\sum_{i=1}^{n} \varphi x_i + b)$$

it follows that f is an n-group with unit a.

The converse part of the theorem is obvious since every TS n-group (Q, f) is of the form $f(x_1^n) = \sum_{i=1}^n x_i + b$, where $(Q, +)$ is a boolean group, $b \in Q$ ([3]). □

Theorem 3. *There exists a completely reducible TS n-loop of order q iff $q = 2^k$, $k \in N$.*

Proof. Since every boolean group is of order 2^k, $k \in N$, from the proof of Theorem 2 it follows that every completely reducible TS n-loop is of that order.

For every $q = 2^k$, $k \in N$ there exists a boolean group $(Q, +)$ of order q. If n is even, then (Q, f), where $f(x_1^n) = \sum_{i=1}^n x_i + b$, is completely reducible TS n-loop with the unit b. If n is odd, then (Q, f), where $f(x_1^n) = \sum_{i=1}^n x_i$, is a completely reducible TS n-loop such that every element of Q is a unit of (Q, f). □

Theorem 4. *For every $q \equiv 2,4 \pmod 6$, $q \neq 2^k$, there exists TS 3-loop of order q which is not reducible.*

Proof. Every Steiner quadruple system (SQS) of order q is equivalent to a TS 3-loop of order q. But SQSs of order q exist iff $q \equiv 2,4 \pmod 6$, and by Theorem 3 reducible TS 3-loops of order q exist iff $q = 2^k$, $k \in N$. Hence every TS 3-loop of order $q \equiv 2,4 \pmod 6$, $q \neq 2^k$ is not reducible. □

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REZIME

O SVODLJIVOSTI TOTALNO SIMETRIČNIH n-KVAZIGRUPA

n-kvazigrupa je potpuno svodljiva ako se može prikazati pomoću $n-1$ binarne kvazigrupe. U ovom radu razmatrana je potpuna svodljivost totalno simetričnih (TS) n-kvazigrupa i n-lupa, $n \geq 3$. Dokazano je da za svaku potpuno svodljivu n-kvazigrupu (Q,f) postoji abelova grupa $(Q,+)$, permutacija φ skupa Q i $b \in Q$ tako da je za svako $x_1^n \in Q$ $f(x_1^n) = \varphi^{-1}(-\sum_{i=1}^n \varphi x_i + b)$. Takodje je dokazano da je svaka TS n-lupa (Q,f) n-grupa sa jedinicom ako i samo ako je (Q,f) potpuno svodljiva (u [3] je naveden pogrešan dokaz ove teoreme). Dokazano je da postoje potpuno svodljive TS n-lupe reda q ako i samo ako je $q = 2^k$, $k \in N$. Jedna posledica ovoga je da za svako $q \equiv 2,4 \ (mod\ 6)$, $q \neq 2^k$, postoji TS 3-lupa reda q koja nije potpuno svodljiva.

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