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ON WEAK CONGRUENCE ALGEBRAS

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Abstract

We give necesary and sufficient conditions for a weak partial congruence algebra (of a given algebra) to be an algebra, so-called weak congruence algebra. We also examine some properties of a weak congruence algebra as well as some applications of the obtained results on some classes of algebras.

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1. Let $\mathcal{A} = (A, F)$ be an algebra and $C \subseteq A$ the set of its constants (if $C = \emptyset$, then, we consider the empty set as a subalgebra of $\mathcal{A}$).

A weak congruence relation ρ on A ([4]) is a symmetric, transitive binary relation on A, satisfying the usual substitution property (if $f \in F_n \subseteq F$, $n \geq 1$, and $x_i \rho y_i, x_i, y_i \in A$, $i = 1, ..., n$ then $f(x_i, ..., x_n) \rho f(y_i, ..., y_n)$), and a weak reflexivity: if $c \in C$, then $c \rho c$.

The notion here will be as follows:

$C_\omega(\mathcal{A})$ is the set of all weak congruences on an algebra $\mathcal{A}$;

$S(\mathcal{A})$ is the set of all subalgebra of $\mathcal{A}$;

$C(\mathcal{A})$ is the set of all congruence on $\mathcal{A}$;

$C(\mathcal{B})$ is the set of all congruences on $\mathcal{B}$, where $\mathcal{B}$ is a subalgebra of $\mathcal{A}$;

$C_\omega(A)$ coincides with the lattice of all congruences on subalgebras of A (under set inclusion), and $(C_\omega(A), \wedge, \vee)$ is the algebraic lattice (see [4]). Moreover ([4]), $C(A)$ is a sublattice of $C_\omega(A)$ (as a filter generated by $\Delta = \{(x,x) \mid x \in A\}$), and $S(A)$ is a retract of $C_\omega(A)$ (ideal generated by Δ). Subalgebras are represented in $C_\omega(A)$ by diagonal relations ($B \in S(A)$ which correspond to $d_{B^2} = \{(x,x) \mid x \in B\}$). Let A be an algebra. By its weak partial congruence algebra ([3]) we mean partial algebra.

$$K_\omega(A) = (C_\omega(\mathrm{A}), \wedge, \vee, o^{-1}, \Delta, \sigma, \mathrm{A}^2),$$

where

$$(C_\omega(\mathrm{A}), \wedge, \vee) = C_\omega(A),$$

$o-$ is the composition of relations (binary partial operation on $C_\omega(\mathrm{A})$,

$^{-1}-$ is the inverse operation for relations, $\sigma-$ is the diagonal relation corresponding to the least subalgebra of $S(A)$. $K_\omega(A)$ gives more informations about A than lattice $C_\omega(A)$, ([3]).

2. In this part we give necessary and sufficient conditions that a weak partial congruence algebra $K_\omega(A)$ (of a given algebra A) is just an algebra, so - called weak congruence algebra.

Theorem 1. *Let $K_\omega(A)$ be an weak partial congruence algebra. Then, for each $\theta, \rho \in C_\omega(A)$:*

$K_\omega(A)$ is a weak congruence algebra iff it holds $\theta \circ \rho = \rho \circ \theta$.

Proof. The same as for (ordinary) ekvivalences and congruences. □

Theorem 2. *A weak partial congruence algebra $K_\omega(A)$ is weak congruence algebra iff the following is satisfied:*

(1) for each $\theta, \rho \in C(A)$ it holds: $\theta \circ \rho = \rho \circ \theta$;

(2) it holds: either $S(A) = \{A\}$ or $S(A) = \{\phi, A\}$.

Proof. ($\Rightarrow$) For $\sigma \in C_\omega(A)$ we have either $\sigma = \Delta$ or ($\sigma \leq \Delta$ and $\sigma \neq \Delta$.

If $\sigma = \Delta$ then $S(A) = \{A\}$ and $C_\omega(A) = C(A)$.

If ($\sigma \leq \Delta$ and $\sigma \neq \Delta$) they for each proper subalgebra $B \in S(A)$ of cardinality ≥ 1 there exist $\theta \in C(B)$ and $\rho \in C(A)$ such that $\theta \circ \rho \notin C_\omega(A)$. Hence $S(A) = \{\phi, A\}$.

($\Leftarrow$) If the conditions (1) and (2) weak satisfied then it follows immediately that $K_\omega(A)$ is a weak congruence algebra. □

Corollary 1. *If $K_\omega(A)$ is weak congruence algebra then $C_\omega(A)$ is modular lattice.*

Remark. The class of algebras which satisfy the condition (2) from Theorem 2 is not closed for direct product.

3. Let us apply previous results on monounary algebras.

An algebra $A = (A, f)$ is a monounary algebra iff f is a unary operation on A. A monounary algebra (A, f) is a finite cycle of leght $n \geq 1$ ($n-$ a nenegative integer) iff $A = \{a_0, a_1, ..., a_{n-1}\}$ and $f(a_i) = a_{i+1}$ for $i = 0, 1, ..., n-2$; $f(a_{n-1}) = a_0$.

Theorem 3. *Let A be a monounary algebra. Then $K_\omega(A)$ is a weak congruence algebra iff A is finite cycle.*

Proof. Propositions follows from the characterization of types of monounary algebras, which have permutable congruence (see T. 1. [2]). Only finite cycles satisfy condition (2) from Theorem 2. □

Let a lattice $L = (L, \wedge, \vee, 0, 1)$. The following holds:

Theorem 4. *$K_\omega(L)$ is a weak congruence algebra iff S is a twoelement chain.*

Proof. By T. 1.5. (see [5]), $C_\omega(L)$ is modular lattice iff L is the twoelement chain. Whence by T.2. and C.3. the results follows directy. □

In order prove the representation theorem it is sufficient to confirm the following hipothetis:

For arbitrary modular algebraic lattice L there exists an algebra A with permutable congruences such that

$$C_\omega(A) \cong L.$$

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REZIME

O SLABIM KONGRUENCIJSKIM ALGEBRAMA

Daju se potrebni i dovoljni uslovi da slaba parcijalna kongruencijska algebra (za datu algebru) bude algebra, tzv. slaba kongruencijska algebra.

Izučavane su neke osobine slabih kongruencijskih algebri. Data je i primena dobijenih rezultata na neke klase algebri.

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