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DOMAIN DECOMPOSITION FOR SPECTRAL APPROXIMATION OF THE LAYER SOLUTION

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Abstract

The boundary layer problems described by the second order differential equation with the small parameter multiplying the highest derivative are considered. For nonselfadjoint, selfadjoint and some turning point problems the domain decomposition is performed, based on the asymptotic behavior of the exact solution. The decomposition depends on the degree of the truncated orthogonal series which approximates the exact solution inside the layer.

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1. Introduction

In this paper we shall consider singularly perturbed problems described by the second order differential equation, which solution displayes either boundary or interior layers. This kind of problems is lately of the great interest for the practical use in technique and sciences such as chemistry, biology and physics. A great number of authors have treated these kind of problems numericly. Most of those methods are descrete ones, adapted to the character of the problem either by the use of some special grids or introducing the fitting factors. The second group of authors has developed continous approximations such as spline functions or similar piecewise approximations. Only a small number of authors have used spectral methods to obtain a continous approximation of singularly perturbed problems. One of the first attempts was made by Caunto in his paper [4], where it was shown that Chebyshev and Legendre truncated orthogonal series can be used to approximate the solution of Helmholtz problem when $\varepsilon n^2 \to \infty$ as $n \to \infty$ (ε is a perturbation parameter and n a degree of spectral approximation). The error is of order $0(\varepsilon^{-1}n^{-4})$, which shows that even modest values of ε require quite a large number of terms in the truncated orthogonal series, and, thus, this result has very small practical value if we are interested in the behavior of the solution inside the layer, where the values change extremly rapidly.

In intention to achieve this the author has tried in several papers to divide the original interval in such a way that the layer subinterval is determined using special "resemblance functions" that were defined according to the type of the singularly perturbed problems that were examined. The similar attempt, based on the theoretical layer length was recently made by Heinrichs in [6].

The results presented in this paper will determine the domain decomposition which, based on the asymptotic behavior of the solution, enables us to construct a highly accurate low degree polynomial approximation of the solution inside the layer. Such an approximation, as a continous function, represents an easily applicitable way for solving singularly perturbed problems, and at this point of view has certain advantages to various discrete methods which can give us only a limited number of points inside the layer.

The main result is that the domain decomposition will be carried out in such a way, that, according to the asymptotic behavior of the solution, we can use one general "resemblance function" for all the cases that the author has examined earlier, as well as for the multiple turning point problems. The obtained domain decomposition depends only on the degree of the truncated orthogonal series and is of order $0(n)$ for all the examined cases (selfadjoint, nonselfadjoint and turning point problems). Compared to the domain decomposition performed by Heinrichs in [6], where the decomposition is fixed, the approximation presented in this paper gives better results when low degree spectral series is used.

In the first section of this paper we shall give some assumptions and some very well known results related to our problem. In the second section we shall give the idea of the construction of the approximate solution, based on the asymptotic behavior. In the third section we shall define the "resemblance function" which will unable us to obtain the domain decomposition addapted to the polynomial approximation of the layer solution. Finaly we shall illustrate the theoretical results by numerical examples.

2. Behavior of the solution

We shall consider the following singularly perturbed problem

$$(2.1) \qquad Ly \equiv -\varepsilon^2 y''(x) + a(x)y'(x) + e(x)y(x) = f(x) \quad x \in [-1,1]$$

$$(2.2) \qquad By \equiv (y(-1), y(1)) = (A, B)$$

where $\varepsilon > 0$ is a small perturbation parameter.

Case I If

a) $a(x) \geq \alpha > 0$, $e(x) \geq \beta$, $\alpha^2 + 4\varepsilon^2\beta > 0$

b) $a(x) \geq \alpha > 0$, $e(x) - a'(x) \geq \gamma$, $\alpha^2 + 4\varepsilon^2\gamma > 0$

we have a nonselfadjoint problem with the unique solution $y(x) \in C^2[-1,1]$ which displayes one boundary layer at the point $x = 1$, of the length $0(\varepsilon^2)$. For the solution $y_r(x)$ of the reduced problem

$$(2.3) \qquad a(x)y_r'(x) + e(x)y_r(x) = f(x), y_r(-1) = A$$

the following estimate holds

$$(2.4) \qquad |y(x) - y_r(x)| \leq C \cdot (\varepsilon^2 + e^{\frac{(x-1)a(1)}{\varepsilon^2}}), x \in [-1,1).$$

Throughout the paper C will denote an arbitrary constant independent of x and ε.

Case II If $a(x) \equiv 0$, $e(x) \geq K^2 > 0$, $K \in R$
we have a selfadjoint problem with the unique solution $y(x) \in C^2[-1,1]$ which displayes two boundary layers of the length $0(\varepsilon)$. For the solution $y_r(x)$ of the reduced problem

$$(2.5) \qquad e(x)y_r(x) = f(x), x \in (-1,1)$$

the following estimate holds

$$|y(x) - y_r(x)| \leq C \cdot (\varepsilon^2 + e^{\frac{K(1+x)}{\varepsilon}} + e^{\frac{K(1-x)}{\varepsilon}}). \tag{2.6}$$

Case III

If 1) $a(x) \equiv -xb(x)$, sign$b(x)$ =const, $x \in [-1,1]$ we have single turning point problem, and

if 2) $a(x) \equiv -x^k b(x)$, $k \geq 2$, sign$b(x)$ =const, $x \in [-1,1]$ we have a multiple turning point problem.

If a) $b(x) \leq -\delta < 0$ and $\frac{e(0)}{b(0)} \neq m$, $m = 0,1,2,\ldots$ the solution tends to zero for $x \in (-1,1)$ and we have two boundary layers of the length $0(\varepsilon^2)$.

If b) $b(x) \geq \delta > 0$, $e(0) = 0$
we have an interior layer at the point $x = 0$, i.e. the shock layer of the length $0(\varepsilon)$. For the solutions $y_e(x)$ and $y_r(x)$ of the left and right reduced problems

$$-xb(x)y_e'(x) + e(x)y_e(x) = f(x) \quad y_e(-1) = A \tag{2.7}$$

$$-xb(x)y_r'(x) + e(x)y_r(x) = f(x) \quad y_r(1) = B \tag{2.8}$$

it holds that

$$y(0) \to \frac{y(-1) + y(1)}{2} = \frac{y_e(0) + y_r(0)}{2} = \frac{A+B}{2} \tag{2.9}$$

and

$$y(x) = \begin{cases} y_e(x) + 0(\varepsilon^2) & -1 \leq x < 0 \\ \frac{A+B}{2} & x = 0 \\ y_r(x) + 0(\varepsilon^2) & 0 < x \leq 1. \end{cases} \tag{2.10}$$

3. Approximation of the solution

The first step is to make use of the reduced solution in representing the exact solution of the considered problems. Thus, we have in

Case I The exact solution is of the form

$$y(x) = y_r(x) + u(x) \tag{3.1}$$

where $u(x)$ will be approximuted by

$$u(x) \approx \begin{cases} 0 & x \in [-1, 1, -c\varepsilon^2] \\ v(x) & x \in [1 - c\varepsilon^2, 1]. \end{cases} \tag{3.2}$$

The function $v(x)$ must be the solution of the following problem

$$Lv(x) = \varepsilon^2 y_r''(x), x \in [1 - c\varepsilon^2, 1]; v(x_0) = 0, v(1) = B - y_r(1) = \\ = B_1, x_0 = 1 - c\varepsilon^2. \tag{3.3}$$

Case II The exact solution is of the form

$$y(x) = y_r(x) + u(x) + w(x) \tag{3.4}$$

where $u(x)$ and $w(x)$ are approximated by

$$u(x) \approx \begin{cases} 0 & x \in [-1, 1 - c\varepsilon] \\ v(x) & x \in [1 - c\varepsilon, 1] \end{cases} ;$$
$$w(x) \approx \begin{cases} v_1(x) & x \in [-1, -1 + c\varepsilon] \\ 0 & x \in [-1 + c\varepsilon, 1] \end{cases} . \tag{3.5}$$

The function $v(x)$ (and similarly $v_1(x)$) must be the solution of the following problem

$$Lv(x) = \varepsilon^2 y_r''(x)\; x \in [1 - c\varepsilon, 1], v(x_0) = 0, v(1) = \\ = B - y_r(1) = B_1, x_0 = 1 - c\varepsilon. \tag{3.6}$$

Case IIIa) We shall approximate the exact solution by

$$y(x) \approx \begin{cases} u(x) & x \in [-1, -1 + c\varepsilon^2] \\ 0 & x \in [-1 + c\varepsilon^2; 1 - c\varepsilon^2] \\ v(x) & x \in [1 - c\varepsilon^2, 1]. \end{cases} \tag{3.7}$$

The function $v(x)$ (and similarly $u(x)$) must satisfy

$$Lv(x) = f(x), x \in [1 - c\varepsilon^2, 1], v(x_0) = 0, \tag{3.8}$$

$$v(1) = B = B_1,\; x_0 = 1 - c\varepsilon^2.$$

Case IIIb) We shall represent the exact solution in the form

$$y(x) = \begin{cases} y_e(x) + u(x) & x \in [-1, 0] \\ y_r(x) + w(x) & x \in [0, 1], \end{cases} \tag{3.9}$$

where $w(x)$ (and similarly $u(x)$) is approximated by

$$w(x) \approx \begin{cases} v(x) & x \in [0, c\varepsilon] \\ 0 & x \in [c\varepsilon, 1]. \end{cases} \tag{3.10}$$

The function $v(x)$ must be the solution of the problem

$$Lv(x) = \varepsilon^2 y_r''(x) x \in [0, c\varepsilon], \tag{3.11}$$

$$v(0) = \frac{A - B}{2} = B_1, v(x_0) = 0, x_0 = c\varepsilon.$$

The idea in all cases is to approximate the function $v(x)$ by a low degree truncated orthogonal series upon the layer subinterval. The accuracy of such an approximation vitaly depends on the suitable domain decomposition determined by the choice of parameter c. This parameter must be determined by the use of "resemblance function" which is defined as:

Definition 1. *The resemblance function is the n-th degree polynomial $p(x)$ such that*

a) $p(x_0) = 0$ *is the* $\min p(x)$ *if* $B_1 > 0$*, and* $\max p(x)$ *if* $B_1 < 0$.

b) $p(x)$ *is concave if* $B_1 > 0$*, and convex if* $B_1 < 0$.

c) $p(1) = B_1$ *(in cases I, II and IIIa) and* $p(0) = B_1$ *in case IIIb).*

Now, we can easily prove the following

Theorem 1. *The resemblance function is*

$$p(x) = B_1 \cdot \left(\frac{x + c\varepsilon^2 - 1}{c\varepsilon^2}\right)^n \text{ in case I and case IIIa)} \tag{3.12}$$

$$p(x) = B_1 \cdot \left(\frac{x + c\varepsilon - 1}{c\varepsilon}\right)^n \text{ in case II and} \tag{3.13}$$

$$p(x) = B_1 \cdot \left(1 - \frac{x}{c\varepsilon}\right)^n \text{ in case IIIb).} \tag{3.14}$$

Proof. The proof will be carried out for (3.12). The proves for the two other formulas is similar.

We have to verify that the expression (3.12) is a resemblance function according to the previous definition. Obviously it is the n-th degree polynomial. The first derivative

$$p'(x) = \frac{nB_1}{c\varepsilon^2} \cdot \left(\frac{x + c\varepsilon^2 - 1}{c\varepsilon^2}\right)^{n-1} = 0$$

for $x = 1 - c\varepsilon^2$ which, according to (3.3) and (3.8), gives $p'(x_0) = 0$. As $p'(x) > 0$, $x \in (x_0, 1)$ for $B_1 > 0$ we conclude that $p(x_0)$ is the minimum for $p(x)$, and similarely $p(x_0)$ is the maximum for $p(x)$ if $B_1 < 0$.

The second derivative

$$p''(x) = \frac{n(n-1)B_1}{c^2\varepsilon^4}\left(\frac{x + c\varepsilon^2 - 1}{c\varepsilon^2}\right)^{n-2}$$

satisfies sgn$p''(x)$ =sgnB_1, which means that $p(x)$ is concave if $B_1 > 0$ and convex if $B_1 < 0$. Finaly we can easily see that $p(1) = B_1$ □.

We shall determine the parameter c as the function of degree n of the chosen spectral approximation from the request that $p(x)$ has to satisfy the differential equation in (3.3) in the neighbourhood of the layer point in the cases I, II and IIIb) and (3.8) in the case IIIa). This gives us the following

Theorem 2. *The parameter $c = c(n)$ is given by the following expressions:*

$$(3.15)\quad c = \frac{B_1 a(1)n - \sqrt{a^2(1)n^2B_1^2 - 4n(n-1)B_1(\varepsilon^4 y_r''(1) - B_1\varepsilon^2 e(1))}}{2(\varepsilon^4 y_r''(1) - B_1\varepsilon^2 e(1))} \approx$$

$$\approx \frac{n-1}{a(1)}$$

in case I,

$$(3.16)\qquad c = \sqrt{\frac{n(n-1)B_1}{e(1)B_1 - \varepsilon^2 y_r''(1)}} \approx \sqrt{\frac{n(n-1)}{e(1)}}$$

in case II

$$(3.17)\qquad c = \frac{nb(1) - \sqrt{n^2b^2(1) + 2n(n-1)e(1)}}{2e(1)} \approx -\frac{n-1}{b(1)}$$

in case IIIa) and

$$c = \sqrt{\frac{n(n-1)B_1}{B_1 b(0) - \varepsilon^2 y_r''(0)}} \approx \sqrt{\frac{n(n-1)}{b(0)}} \tag{3.18}$$

in case IIIb).

Proof. We shall carry out the proof in the case I. The other three cases can be proved in a similar way.
If we introduce (3.12) into the differential equation in (3.3) we obtain

$$-\frac{B_1 n(n-1)}{\varepsilon^2 c^2}\left(\frac{x+c\varepsilon^2-1}{c\varepsilon^2}\right)^{n-2} + a(x)\frac{nB_1}{c\varepsilon^2}\left(\frac{x+c\varepsilon^2-1}{c\varepsilon^2}\right)^{n-1} +$$

$$e(x)B_1\left(\frac{x+c\varepsilon^2-1}{c\varepsilon^2}\right)^{n} = \varepsilon^2 y_r''(x).$$

For $x=1$ this gives us the equation

$$\varepsilon^2(\varepsilon^2 y_r''(1) - e(1)B_1)c^2 - a(1)nB_1 c + B_1 n(n-1) = 0.$$

Its solution c from the interval $(x_0, 1)$ is given by (3.15). The approximate value in (3.15) is obtained by the use of first two terms in McLaurines expansion of the square root. □

4. Numerical examples

In this section we shall give a numerical example for each of the cases examined. We shall evaluate the approximate solution in several points from the layer interval, chosen in such a way that the values of the solution change approximatly equaly. For the spectral approximation we shall use the Chebyshev orthogonal basis and we shall evaluate the exact error as well as the error estimate obtained by constructing the upper and lower solution, based on the principle of inverse monotonicity, presented in [1], [2] and [3].

Case I The layer subinterval for the problem

$$-\varepsilon^2 y''(x) + \frac{2+4\varepsilon^2-2\varepsilon^2 x}{(2-x)^2}y'(x) = \frac{\varepsilon^2\pi^2}{(2-x)^4}\cos\frac{\pi(1-x)}{(2-x)} + \frac{2\pi}{(2-x)^4}\sin\frac{\pi(1-x)}{2-x}$$

$$x \in [0,1] \qquad y(0)=0 \qquad y(1)=0$$

determined by the use of (3.15), for $n = 8$ and $\varepsilon = 0.001$ $[0.9999965, 1]$. The approximate solution, the exact error and the error estimate are given in the following table:

x	y	$\lvert y - y_n \rvert$	$d(x)$
0.999998	0.982	$8.14 \cdot 10^{-4}$	$8.15 \cdot 10^{-4}$
0.999999	0.865	$8.27 \cdot 10^{-4}$	$8.95 \cdot 10^{-4}$
0.9999993	0.753	$7.96 \cdot 10^{-4}$	$7.81 \cdot 10^{-4}$
0.9999994	0.699	$7.48 \cdot 10^{-4}$	$6.73 \cdot 10^{-4}$
0.9999996	0.551	$8.11 \cdot 10^{-4}$	$7.37 \cdot 10^{-4}$
0.9999997	0.451	$4.97 \cdot 10^{-4}$	$5.52 \cdot 10^{-4}$
0.9999998	0.330	$8.43 \cdot 10^{-4}$	$7.87 \cdot 10^{-4}$
0.99999985	0.259	$2.90 \cdot 10^{-4}$	$6.67 \cdot 10^{-4}$
0.9999999	0.181	$3.21 \cdot 10^{-4}$	$4.72 \cdot 10^{-4}$
0.99999999	0.020	$4.89 \cdot 10^{-4}$	$3.74 \cdot 10^{-4}$

Case II The left layer subinterval for the problem

$$-\varepsilon^2 y''(x) + \frac{1-\varepsilon}{(2-x)^2} y(x) = \frac{(1-\varepsilon)(x-1)}{(2-x)^2} \quad x \in [0,1], y(0) = 0, y(1) = 0$$

determined by the use of (3.16) for $n = 12$ and $\varepsilon = 0.00001$ is $[0, 0.00023]$. The approximate solution, the exact error and the error estimate are given in the following table:

x	y	$\lvert y - y_n \rvert$	$d(x)$
0.000001	−0.04876	$1.04 \cdot 10^{-5}$	$5.0 \cdot 10^{-6}$
0.000003	−0.13928	$7.8 \cdot 10^{-6}$	$1.4 \cdot 10^{-5}$
0.000006	−0.25918	$2.5 \cdot 10^{-6}$	$2.4 \cdot 10^{-5}$
0.00001	−0.39346	$5.1 \cdot 10^{-6}$	$3.4 \cdot 10^{-5}$
0.000013	−0.47794	$1.7 \cdot 10^{-6}$	$3.9 \cdot 10^{-5}$
0.00002	−0.63211	$8.2 \cdot 10^{-6}$	$4.2 \cdot 10^{-5}$
0.00003	−0.77685	$2.0 \cdot 10^{-6}$	$3.1 \cdot 10^{-5}$
0.00004	−0.86463	$5.2 \cdot 10^{-6}$	$2.1 \cdot 10^{-5}$
0.0001	−0.99316	$8.0 \cdot 10^{-6}$	$9.2 \cdot 10^{-6}$
0.0002	−0.99975	$1.1 \cdot 10^{-6}$	$1.2 \cdot 10^{-6}$

Case IIIb) The layer subinterval for the problem

$$-\varepsilon^2 y''(x) - xy(x) = 0 \quad x \in [-1, 1], y(-1) = 0, y(1) = 2,$$

determined by the use of (3.17) for $n = 12$ and $\varepsilon = 0.0000001$ is $[-0.000001625, 0.000001625]$. The approximate solution, the exact error and the error estimate for $x > 0$ are given in the following table:

x	y	$\|y - y_n\|$	$d(x)$
0.00000001	1.078	$4.9 \cdot 10^{-4}$	$4.5 \cdot 10^{-4}$
0.00000002	1.159	$2.4 \cdot 10^{-4}$	$4.6 \cdot 10^{-4}$
0.00000003	1.236	$1.4 \cdot 10^{-3}$	$1.7 \cdot 10^{-3}$
0.00000005	1.383	$3.1 \cdot 10^{-3}$	$3.2 \cdot 10^{-3}$
0.00000006	1.451	$3.2 \cdot 10^{-3}$	$3.2 \cdot 10^{-3}$
0.00000008	1.576	$2.2 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$
0.0000001	1.683	$2.3 \cdot 10^{-4}$	$1.2 \cdot 10^{-4}$
0.00000015	1.867	$2.3 \cdot 10^{-3}$	$2.7 \cdot 10^{-3}$
0.0000002	1.954	$1.2 \cdot 10^{-3}$	$9.8 \cdot 10^{-4}$
0.0000003	1.997	$1.7 \cdot 10^{-3}$	$1.8 \cdot 10^{-3}$

If we want to compare these results with the descrete methods combined with the domain decomposition we can see that spectral methods using n-th degree orthogonal polynomials are far more accurate to the approximate solution on n-dimensional grid (see [6]). In those results the domain decomposition is perforemd using c which depends on the perturbation parameter ε. Comparing those results to the results obtained by the technique developed in this paper we can see that the spectral approximation presented here is more accurate for the low degree series.

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REZIME

DEKOMPOZICIJA DOMENA PRI SPEKTRALNOJ APROKSIMACIJI SLOJNOG REŠENJA

Posmatrani su konturni problemi opisani diferencijalnim jednačinama drugog reda sa malim parametrom uz drugi izvod. Za nesamoadjungovan, samoadjungovan problem i neke probleme sa povratnom tačkom izvršena je dekompozicija domena, zasnovana na asimptotskom ponašanju rešenja, u zavisnosti od stepena konačnog ortogonalnog reda kojim se aproksimira tačno rešenje unutar sloja.

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