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ON A FINITE DIFFERENCE ANALOGUE FOR BOUNDARY VALUE PROBLEM

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Abstract

We consider a finite difference analogue for boundary value problem obtained by a five-point difference scheme on an uniform mesh. For the matrix arising from this analogue an explicit inverse matrix is derived and some of its properties are proved.

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1. Introduction

Let us consider the following boundary value problem:

$$(1)\qquad \begin{gathered} -u'' + q(x)u(x) = f(x), \quad x \in I = [0,1], \\ u(0) = \alpha, \quad u(1) = \beta. \end{gathered}$$

Here, α and β are given real constants, and $f(x)$, $q(x)$ are given real continuous functions on I, with

$$q(x) \geq 0.$$

The numerical solution of two-point boundary value problem (1) is most commonly obtained by finite difference methods. We place a uniform mesh of size

$$h = \frac{1}{n+1}$$

on I, and denote the mesh points of the discrete problem by

$$x_i = ih, \quad i = 0, 1, ..., n+1. \tag{2}$$

The best-known method for deriving finite difference approximation to (1) is based on finite Taylor's series expansions of the solution $u(x)$ of (1). A set of difference schemes for numerical solution of two-point boundary value problem one can find in [1]. Here, we shall consider a modification of one of these schemes, and derive some properties of the corresponding matrix.

Let us assume that the unique solution $u(x)$ of (1) is of class $C^{(6)}(I)$. Denoting $u(x_i)$ by u_i we have, see [1],

$$\begin{aligned} -u_i'' &= h^{-2}\left(-u_{i-1} + 2u_i - u_{i+1}\right) + r_i, \\ r_i &= \tfrac{h^2}{12} u^{(4)}(\theta_i), \qquad \theta_i \in (x_{i-1}, x_{i+1}), \end{aligned} \tag{3}$$

and

$$\begin{aligned} -u_i'' &= \tfrac{h^{-2}}{12}\left(u_{i-2} - 16u_{i-1} + 30u_i - 16u_{i+1} + u_{i+2}\right) + r_i, \\ r_i &= \tfrac{h^4}{45}\left(2u^{(6)}(\tau_i^1) - 8u^{(6)}(\tau_i^2)\right), \\ &\tau_i^1 \in (x_{i-1}, x_{i+1}), \tau_i^2 \in (x_{i-2}, x_{i+2}). \end{aligned} \tag{4}$$

In order to form a discrete analogue for (1) one can use (3) at the points x_1 and x_n, and (4) at the points $x_i, i = 2, 3, ..., n-1$, see [1]. Here, we have used

$$\begin{aligned} -u_1'' &= \tfrac{h^{-2}}{12}\left(-14u_0 + 29u_1 - 16u_2 + u_3\right) + \tfrac{u_0''}{12} + r_1, \\ r_1 &= \tfrac{h^2}{144}\left(15u^{(4)}(\tau_1^1) - 14u^{(4)}(\tau_1^2)\right), \\ &\tau_1^1 \in (x_0, x_2), \tau_1^2 \in (x_1, x_3). \end{aligned} \tag{5}$$

at x_1, and analogously at x_n,

$$(6)\quad \begin{aligned} -u_n'' &= \frac{h^{-2}}{12}\left(u_{n-2} - 16u_{n-1} + 29u_n - 14u_{n+1}\right) + \frac{u_{n+1}''}{12} + r_n,\\ r_n &= \frac{h^2}{144}\left(15u^{(4)}(\tau_n^1) - 14u^{(4)}(\tau_n^2)\right),\\ &\tau_n^1 \in (x_{n-1}, x_{n+1}), \tau_n^2 \in (x_{n-2}, x_{n-1}). \end{aligned}$$

Using (1), (4), (5), (6), and $u_0 = \alpha$, $u_{n+1} = \beta$, we obtain n equations for n unknowns. In matrix notation, we can write this in the form

$$(7)\qquad Bu = d + r(u),$$

where B is $n \times n$ matrix, and u, d and $r(u)$ are column vectors, given by

$$B = \frac{h^{-2}}{12}\begin{bmatrix} 29+q_1h^2 & -16 & 1 & & & \\ -16 & 30+q_2h^2 & -16 & 1 & & \\ 1 & -16 & 30+q_3h^2 & -16 & 1 & \\ & \cdots & \cdots & \cdots & \cdots & \\ & \cdots & \cdots & \cdots & \cdots & \\ & 1 & -16 & 30+q_{n-2}h^2 & -16 & 1 \\ & & 1 & -16 & 30+q_{n-1}h^2 & -16 \\ & & & 1 & -16 & 29+q_nh^2 \end{bmatrix},$$

$$u = \left[u_1, u_2, u_3, \ldots, u_{n-2}, u_{n-1}, u_n\right]^{\top},$$

$$d = \begin{bmatrix} f_1 + \frac{q_0\alpha}{12} - f_0 + \frac{14\alpha}{12h^2} \\ f_2 - \frac{\alpha}{12h^2} \\ f_3 \\ \cdot \\ \cdot \\ \cdot \\ f_{n-2} \\ f_{n-1} - \frac{\beta}{12h^2} \\ f_n + \frac{q_{n+1}\beta}{12} - f_{n+1} + \frac{14\beta}{12h^2} \end{bmatrix}, \quad r = \begin{bmatrix} \frac{h^2}{144}\left(15u^{(4)}(\tau_1^1) - 14u^{(4)}(\tau_1^2)\right) \\ \frac{h^4}{45}\left(2u^{(6)}(\tau_2^1) - 8u^{(6)}(\tau_2^2)\right) \\ \frac{h^4}{45}\left(2u^{(6)}(\tau_3^1) - 8u^{(6)}(\tau_3^2)\right) \\ \cdot \\ \cdot \\ \cdot \\ \frac{h^4}{45}\left(2u^{(6)}(\tau_{n-2}^1) - 8u^{(6)}(\tau_{n-2}^2)\right) \\ \frac{h^4}{45}\left(2u^{(6)}(\tau_{n-1}^1) - 8u^{(6)}(\tau_{n-1}^2)\right) \\ \frac{h^2}{144}\left(15u^{(4)}(\tau_n^1) - 14u^{(4)}(\tau_n^2)\right) \end{bmatrix}.$$

Let us define the solution vector z of

$$Bz = d \tag{8}$$

as our discrete approximation of the solution $u(x)$ for problem (1).

2. Some properties of discrete analogue

A matrix A is called inverse monotone if A has an inverse $A^{-1} \geq 0$, see [1].

Theorem 1. *Let B_0 be the matrix B in the case $q(x) = 0, x \in I$. Then,*

a) Matrix B_0 is inverse monotone.

b) $\|B_0^{-1}\|_\infty \leq \frac{1}{8}$.

c) The inverse of the matrix B_0 is given explicitly by $B_0^{-1} = [b_{i,j}]$ where $b_{i,j} = b_{j,i}$ and

$$b_{i,j} = h^2 \left(\frac{j(n+1-i)}{n+1} - \frac{d_{j-1} d_{n-i}}{d_n} \right), \quad i \geq j,$$

$$d_i = \frac{sinh((i+1)\theta)}{sinh(\theta)}, \qquad \theta = arccosh(7).$$

Proof. It is easy to see that

$$B_0 = \frac{h^{-2}}{12} \left(A^2 + 12A \right),$$

where

$$A = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & \cdots & \cdots & \cdots & & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

So, we have

$$B_0 = 12h^2(A + 12E)^{-1} A^{-1},$$

where E is identity matrix. Since A and $A+12E$ are inverse monotone matrices, see [5], [1], it folows that B_0 is inverse monotone matrix too.

From the inequlities, see [5],

$$\|(A+12E)^{-1}\|_\infty \le \frac{1}{12}, \qquad \|h^2A^{-1}\| \le \frac{1}{8}$$

we have b).

For the matrix

$$J(c) = \begin{bmatrix} 2c-1 & 2-c & -1 & & & & & \\ 2-c & 2c-2 & 2-c & -1 & & & & \\ -1 & 2-c & 2c-2 & 2-c & -1 & & & \\ & \cdots & \cdots & \cdots & \cdots & & & \\ & \cdots & \cdots & \cdots & \cdots & & & \\ & -1 & 2-c & 2c-2 & 2-c & -1 & & \\ & & -1 & 2-c & 2c-2 & 2-c & & \\ & & & -1 & 2-c & 2c-1 & & \end{bmatrix},$$

exact inverse is given in [4]. It is obvious that

$$B_0 = -\frac{h^{-2}}{12}J(-14),$$

and using the results from [4], we obtain our statement.

□

Let us multiply first and last equation of the system (7) with h^2. Then we can write it as

$$B_h = d_h + r_h(u), \tag{9}$$

where

$$B_h = \frac{h^{-2}}{12}\begin{bmatrix} 29h^2 & -16h^2 & h^2 & & & & \\ -16 & 30 & -16 & 1 & & & \\ 1 & -16 & 30 & -16 & 1 & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & & \\ & 1 & -16 & 30 & -16 & 1 & \\ & & 1 & -16 & 30 & -16 & \\ & & & h^2 & -16h^2 & 29h^2 & \end{bmatrix},$$

$d_h = diag(h^2, 1, \ldots, 1, h^2)d, r_h(u) = diag(h^2, 1, \ldots, 1, h^2)r(u)$. Obviously, systems (8) and (9) are equivalent.

Theorem 2. *Let $u(x)$ be the solution of (1), where $Q(x) = 0$, and let its discrete approximation z be defined by (8). If $u(x) \in C^4(I)$ then*

$$\|u - z\|_\infty \le Mh^4$$

with some constant M independent of h.

Proof. From systems (7) and (9) we have

$$u - z = B_h^{-1} r_h(u)$$

i.e.

$$\|u - z\|_\infty \le M\|B_h^{-1}\|_\infty h^4,$$

since $\|r_h(u)\|_\infty \le Mh^4$

In order to obtain an estimation for $\|B_h^{-1}\|_\infty$ let us note that if

$$B_0^{-1} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1,n-1} & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2,n-1} & b_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n-1,1} & b_{n-1,2} & \ldots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n1} & b_{n2} & \ldots & b_{n-1,n-1} & b_{nn} \end{bmatrix},$$

then

$$B_h^{-1} = \begin{bmatrix} h^{-2}b_{11} & b_{12} & \ldots & b_{1,n-1} & h^{-2}b_{1n} \\ h^{-2}b_{21} & b_{22} & \ldots & b_{2,n-1} & h^{-2}b_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ h^{-2}b_{n-1,1} & b_{n-1,2} & \ldots & b_{n-1,n-1} & h^{-2}b_{n-1,n} \\ h^{-2}b_{n1} & b_{n2} & \ldots & b_{n-1,n-1} & h^{-2}b_{nn} \end{bmatrix},$$

and

$$B_h^{-1} - B_0^{-1} = \left(h^{-2} - 1\right) \begin{bmatrix} b_{11} & 0 & \ldots & 0 & b_{1n} \\ b_{21} & 0 & \ldots & 0 & b_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n-1,1} & 0 & \ldots & 0 & b_{n-1,n} \\ b_{n1} & 0 & \ldots & 0 & b_{nn} \end{bmatrix}.$$

Since

$$0 \le b_{ij} \le \frac{i(n+1-j)h^2}{n+1}, \quad i \le j, \quad b_{ji} = b_{ij},$$

we have

$$\|B_h^{-1} - B_0^{-1}\|_\infty = \frac{1-h^2}{h^2} \max_{i \le n}(b_{i1} + b_{in}) = 1 - h^2.$$

Now, from

$$|\|B_h^{-1}\|_\infty - \|B_0^{-1}\|_\infty| \le \|B_h^{-1} - B_0^{-1}\|_\infty \le 1 - h^2$$

it follows

$$\|B_h^{-1}\|_\infty \le \|B_0^{-1}\|_\infty + 1 - h^2 \le \frac{1}{8} + 1 - h^2 \le M.$$

□

In the case $q(x) \ne 0, x \in I$ and $q(x) \in C(I)$, matrix B can be written as $B_0 + Q$, where

$$Q = diag(29 + q_1 h^2, 30 + q_2 h^2, \dots, 30 + q_{n-1} h^2, 29 + q_n h^2).$$

As $q(x)$ is continuous function, there exists Q^* with the property $Q^* \ge |q(x)|, x \in I$.

If $Q^* h^2 < 8$ the previous theorem is valid in the case $q(x) \ne 0$, because of

$$\|B_0^{-1} Q\|_\infty < \frac{1}{8}\|Q\|_\infty \le \frac{1}{8} Q^* h^2 < 1,$$

so

$$\|B^{-1}\|_\infty = \|(B_0 + Q)^{-1}\|_\infty = \|(E + B_0^{-1}Q)^{-1} B_0^{-1}\|_\infty \le$$

$$\frac{8}{8 - Q^* h^2}\frac{1}{8} = \frac{1}{8 - Q^* h^2} \le M.$$

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REZIME

O JEDNOM DIFERENCNOM ANALOGONU ZA KONTURNI PROBLEM

Posmatra se linearni deo diskretnog analogona konturnog problema dobijen pomoću petotačkaste diferencne aproksimacije drugog izvoda na ekvidistanoj mreži. Za dobijenu matricu odredjena je eksplicitno inverzna matrica i date su neke njene osobine.

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