Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 24, 1 (1994), 389-396

Review of Research
Faculty of Science
Mathematics Series

ON A NONLINEAR GENERALIZATION OF A MATRIX DIAGONAL DOMINANCE WITH APPLICATION TO THE AOR METHOD

Dragoslav Herceg, Ljiljana Cvetković
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

The problem of solving a nonlinear system $F\mathbf{x} = 0$, $F : R^n \to R^n$, where mapping F has some special properties, is present in a number of areas such as optimization problems, finite difference scheme for numerical solution of differential equations. In this paper, a case when F is strictly diagonally dominant (SDD) is considered. In order to solve such problem, an modification of the nonlinear AOR method is proposed and its global convergence is shown.

AMS Mathematics Subject Classification (1991): 65H10
Key words and phrases: Nonlinear systems, iterative methods, relaxation methods, convergence area.

1. Introduction

In this paper an iterative method for solving

(1) $$F\mathbf{x} = 0, \quad F : D \subset R^n \to R^n$$

$$F\mathbf{x} = [f_1(\mathbf{x}), ..., f_n(\mathbf{x})]^T, \quad f_i : D \subset R^n \to R, \; i = 1, ..., n$$

is considered. The iterative method has the following form

$$\mathbf{x}^{k+1} = H\mathbf{x}^k \quad k = 0, 1, \dots \; . \tag{2}$$

In the case when F is a linear mapping, i.e. $F\mathbf{x} = A\mathbf{x} - \mathbf{b}$, where $A = [a_{ij}] \in R^{n\times n}$, $\mathbf{b} \in R^n$ there are many iterative methods present. One of them is a two-parameter AOR method [3]. Among other things, the convergence of AOR procedure is shown in [2], [4], providing matrix A is strictly diagonally dominant (SDD) and ω, $\sigma \in O_{\omega,\sigma}$, where the set $O_{\omega,\sigma}$ contains the unit square.

A generalization of results from linear to nonlinear case is presented in this paper. For this purpose, in section 2 definition of the SDD mapping is given as a generalization of this property for linear mapping. Section 3 contains a modification of the nonlinear AOR method and a corresponding theorem of global convergence of the method for solving system (1), in the case of SDD mapping.

2. Definitions and preliminary results

We recall the definitions of certain clases of matrices that will play a role in this paper.

Definition 1. *Matrix $A \in R^{n\times n}$ is strictly diagonally dominant (SDD) matrix if*

$$|a_{ii}| > \sum_{i\neq j} |a_{ij}|, \quad i = 1, \dots, n.$$

Generalization of this property to nonlinear mappings is given through the following definitions:

Definition 2. *A function $f : D \subset R^n \to R$ is strictly diagonally dominant (SDD) on D with respect to the ith variable if*

$$\forall \mathbf{x}, \mathbf{y} \in D, \quad \mathbf{x} \neq \mathbf{y} \;\wedge\; f(\mathbf{x}) = f(\mathbf{y}) \;\Rightarrow\; |x_i - y_i| < ||\mathbf{x} - \mathbf{y}||_\infty.$$

Definition 3. *A mapping $F : D \subset R^n \to R^n$ is strictly diagonally dominant (SDD) on D if for each $i = 1, \dots, n$ function $f_i : D \subset R^n \to R$ is SDD with respect to the ith variable.*

It is easily shown that the matrix A is a SDD matrix if and only if the induced mapping Fx=Ax is an SDD function on R^n. Following result gives sufficient conditions for a mapping F to be an SDD mapping in terms of its derivative. The proof will be omitted (see [7]).

Theorem 1. *Let $F : D \subset R^n \to R^n$ be G-differentiable on the convex set D, and assume that $F'(\mathbf{x})$ is SDD matrix for each $\mathbf{x}$ in D. Then F is a SDD mapping on D. ($F'(\mathbf{x})$ denotes Jacobian's matrix).*

Before we get on to the definition of a concrete iterative procedure, we present a theorem for the general explicit iterative procedure (2) which is necessary for the work to follow. The proof of this theorem can be found in [5].

Theorem 2. *Let $H : D \subset R^n \to R^n$ be a mapping of compact set D into itself and suppose that*

$$||H\mathbf{x} - H\mathbf{y}|| < ||\mathbf{x} - \mathbf{y}|| \quad \mathbf{x}, \mathbf{y} \in D \;\; \mathbf{x} \neq \mathbf{y},$$

for some norm. Then H has a unique fixed point in D, and for any $\mathbf{x}^0 \in D$ array $\{\mathbf{x}^k\}_{k=0}^{\infty}$ of iterates converges to this fixed point.

3. Modified AOR algorithm and convergence theorem

The modified AOR (Accelerated Overrelaxation) iteration are given by

for $i = 1$ to n:

1. step: Solve, for x_i,

$$f_i(\bar{x}_1^{k+1}, \bar{x}_2^{k+1}, \ldots, \bar{x}_{i-1}^{k+1}, x_i, x_{i+1}^k, \ldots, x_n^k) = 0 \tag{3}$$

2. step: Set

$$\bar{x}_i^{k+1} = (1 - \sigma_i)x_i^k + \sigma_i x_i \tag{4}$$

3. step: Set

$$x_i^{k+1} = (1 - \omega_i)x_i^k + \omega_i x_i, \tag{5}$$

where, ω_i and σ_i are real parameters.

As the special cases of the method (3) - (5) we have:

if $\sigma_i = 0$ for all i JOR method is obtained;

if $\sigma_i = \omega_i = \omega$ for all i SOR method arise.

It is well known that in a linear case AOR method converges globally if the system matrix A is SDD matrix and $\omega \in (0,1]$, $\sigma \in [0,1]$. The following theorem shows that the similar result holds in the case of a nonlinear SDD mapping.

Theorem 4. *Let $F : D \subset R^n \to R^n$ be an SDD mapping on D which is a convex set. Suppose that for each $\mathbf{x}$ in D and for each $i = 1, ..., n$ the one-dimensional equation*

$$f_i(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_n) = 0$$

has a solution t_i^* with

$$[x_1, ..., x_{i-1}, t_i^*, x_{i+1}, ..., x_n]^T \in D.$$

Then the modified AOR sequences (3) - (5) with $\omega_i \in (0,1]$, $\sigma_i \in [0,1]$, $i = 1, 2, \ldots n$, $k = 0, 1, ...$ are well-defined for any $\mathbf{x}^0 \in D$ and there is an iteration function $H : D \subset R^n \to R^n$ such that:

(*i*) *the method is equivalent with* $\mathbf{x}^{k+1} = H\mathbf{x}^k, \quad k = 0, 1, ...$

(*ii*) $H(D) \subset D$

(*iii*) $||H\mathbf{x} - H\mathbf{y}||_\infty < ||\mathbf{x} - \mathbf{y}||_\infty$ *for every* $\mathbf{x}, \mathbf{y}$ *in* D.

Proof. Let $\mathbf{x} \in D$ be given.

For $k = 1$ by assumption there is t_1^*, which is unique (because SDD mapping is injective), such that

$$f_1(t_1^*, x_2, ..., x_n) = 0$$

and $[t_1^*, x_2, ..., x_n]^T \in D$. Set

(6) $$h_1^\sigma(\mathbf{x}) = (1 - \sigma_1)x_1 + \sigma_1 t_1^* \quad h_1(\mathbf{x}) = (1 - \omega_1)x_1 + \omega_1 t_1^*.$$

D is convex by assumption, so

$$[h_1^\sigma(\mathbf{x}), x_2, ..., x_n]^T,\ [h_1(\mathbf{x}), x_2, ..., x_n]^T \in D.$$

For $k = 1, ..., i-1$ assume that $h_k^\sigma(\mathbf{x})$ and $h_k(\mathbf{x})$ have been defined that

$$(7) \qquad [h_1^\sigma(\mathbf{x}), ..., h_{i-1}^\sigma, x_i, ..., x_n]^T,\ [h_1(\mathbf{x}), ..., h_{i-1}, x_i, ..., x_n]^T, \in D.$$

For $k = i$ there is a unique t_i^* such that

$$f_i(h_1^\sigma(\mathbf{x}), ..., h_{i-1}^\sigma(\mathbf{x}), t_i^*, x_{i+1}, ..., x_n) = 0$$

and we set

$$h_i^\sigma(\mathbf{x}) = (1-\sigma_i)x_i + \sigma_i t_i^* \quad h_i(\mathbf{x}) = (1-\omega_i)x_i + \omega_i t_i^*.$$

D is convex and

$$[h_1^\sigma(\mathbf{x}), ..., h_{i-1}^\sigma(\mathbf{x}), h_i^\sigma(\mathbf{x}), x_{i+1}, ..., x_n]^T \in D,$$

$$[h_1(\mathbf{x}), ..., h_{i-1}(\mathbf{x}), h_i(\mathbf{x}), x_{i+1}, ..., x_n] \in D.$$

In this way, we have defined $H : D \subset R^n \to R^n$ $H = [h_1, ..., h_n]^T$ such that $H(D) \subset D$ and such that for every $\mathbf{x}^0 \in D$ the modified AOR method is well-defined and equivalent with $\mathbf{x}^{k+1} = H\mathbf{x}^k$. Thus we have shown (i) and (ii). In order to prove (iii) we shall use induction again.

Let $\mathbf{x}, \mathbf{y} \in D$ be given.

For $k = 1$: Set $\bar{\mathbf{x}} = [\bar{x}_1, x_2, ..., x_n]^T$, $\bar{\mathbf{y}} = [\bar{y}_1, y_2, ..., y_n]^T$ where $f_1(\bar{\mathbf{x}}) = 0$ and $f_1(\bar{y}) = 0$. From (6) we have

$$|h_1(\mathbf{x}) - h_1(\mathbf{y})| \le (1-\omega_1)|x_1 - y_1| + \omega_1|\bar{x}_1 - \bar{y}_1|.$$

Two cases are discerned:

- If $\bar{x}_1 = \bar{y}_1$ then

$$|h_1(\mathbf{x}) - h_1(\mathbf{y})| \le (1-\omega_1)|x_1 - y_1| < |x_1 - y_1| \le ||\mathbf{x} - \mathbf{y}||_\infty.$$

· If $\bar{x}_1 \neq \bar{y}_1$ then:

$$\bar{\mathbf{x}} \neq \bar{\mathbf{y}} \ \wedge \ f_1(\bar{\mathbf{x}}) = f_1(\bar{\mathbf{y}}) \ \text{ and F is SDD Mapping } \Rightarrow$$

$$\Rightarrow |\bar{x}_1 - \bar{y}_1| < ||\bar{\mathbf{x}} - \bar{\mathbf{y}}||_\infty \leq ||\mathbf{x} - \mathbf{y}||_\infty$$

and

$$|h_1(\mathbf{x}) - h_1(\mathbf{y})| < ||\mathbf{x} - \mathbf{y}||_\infty.$$

For $k = 1, .., i-1$ we assume that

$$|h_k(\mathbf{x}) - h_k(\mathbf{y})| < ||\mathbf{x} - \mathbf{y}||_\infty.$$

For $k = i$: Set $\bar{\mathbf{x}} = [h_1^\sigma(\mathbf{x}), ..., h_{i-1}^\sigma(\mathbf{x}), \bar{x}_i, x_{i+1}, ..., x_n]$, where $f_i(\bar{\mathbf{x}}) = 0$, and similarly for $\bar{\mathbf{y}}$.

$$|h_i(\mathbf{x}) - h_i(\mathbf{y})| \leq (1 - \omega_i)|x_i - y_i| + \omega_i|\bar{x}_i - \bar{y}_i|$$

· If $\bar{x}_i = \bar{y}_i$ then

$$|h_i(\mathbf{x}) - h_i(\mathbf{y})| \leq (1 - \omega_i)|x_i - y_i| < |x_i - y_i| \leq ||\mathbf{x} - \mathbf{y}||_\infty.$$

· If $\bar{x}_i \neq \bar{y}_i$ then

$$\bar{\mathbf{x}} \neq \bar{\mathbf{y}} \ \wedge \ f_i(\bar{\mathbf{x}}) = f_i(\bar{\mathbf{y}}) \ \text{ and F is SDD mapping } \Rightarrow$$

$$\Rightarrow \ |\bar{x}_i - \bar{y}_i| < ||\bar{\mathbf{x}} - \bar{\mathbf{y}}||_\infty \leq ||\mathbf{x} - \mathbf{y}||_\infty.$$

We have derived

$$|h_i(\mathbf{x}) - h_i(\mathbf{y})| < ||\mathbf{x} - \mathbf{y}||_\infty, \quad i = 1, ..., n,$$

i.e. the statement (iii) is proved. □

Now we can prove our main result.

Theorem 5. *Let $F : D \subset R^n \to R^n$ satysfy the hipotheses of Theorem 4 on the convex and compact set D. Then modified AOR method (3)-(5) converges to the unique solution $\mathbf{x}^*$ in D of equation $F\mathbf{x} = 0$, for any $\mathbf{x}^0 \in D$ with $\omega_i \in (0, 1]$, $i = 1, 2, \ldots n$ and $\sigma_i \in [0, 1]$, $i = 1, 2, \ldots n$.*

Proof. On the basis of Theorem 4 we know that all the assumptions of Theorem 3 are satisfied for mapping H and the result follows. □

References

[1] Cvetković, Lj., Herceg, D., Nonlinear AOR method, Z. angew. Math. Mech. 68(1988), 486-487.

[2] Cvetković, Lj., Herceg, D., Convergence theory for AOR method, Journal of Computational Mathematics 8(1990), 128-134.

[3] Hadjidimos, A., Accelerated Overrelaxation Method, Math. Comp. 32(1978), 149-157.

[4] Herceg, D., Cvetković, Lj., An improvement for the area of convergence of the AOR method, Anal. Numer. Theor. Approx. 16(1987), 109-115.

[5] Goebel, K., Kirk, W., Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[6] Krejić, N., Ogrizović, Z., Nonlinear Accelerated Overrelaxation method, IX Conference on Applied Mathematics, (Herceg, D., Cvetković, Lj., eds.), Budva 1994, (in print).

[7] Moré, J.J., Nonlinear Generalizations of Matrix Diagonal Domminance with Application to Gauss-Seidel Iterations, SIAM J. Numer. Anal. 9 (1972), 357-378.

[8] Ortega, J.M., Rheinboldt, W.C., Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.

REZIME

O NELINEARNOJ GENERALIZACIJI MATRIČNE DIJAGONALNE DOMINACIJE SA PRIMENOM NA AOR POSTUPAK

Problem rešavanja nelinearnog sistema $F\mathbf{x} = 0$, $F : R^n \to R^n$, pri čemu preslikavanje F ima neke specijalne osobine, javlja se u brojnim oblastima, kao što su problemi optimizacije, konačno diferencne šeme za rešavanje diferencijalnih jednačina. U ovom radu se razmatra slučaj kada je F strogo dijagonalno dominantno (SDD) preslikavanje. Predložena je jedna modifikacija nelinearnog AOR postupka za rešavanje ovakvih problema i dokazana njena globalna konvergencija.

Received by the editors May 17, 1993