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A FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS IN 2-MENGER SPACES

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Abstract

A fixed point theorem for multivalued C-contraction in 2-Menger spaces is proved.

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1. Introduction

The theory of probabilistic metric spaces is an important part of Stochastic Analysis, and so it is of interest to develop the fixed point theory in such spaces [16].

The first result from the fixed point theory in probabilistic metric spaces is obtained by Sehgal and Bharucha-Reid in [18]. Since then many fixed point theorems for singlevalued and multivalued mappings in probabilistic metric spaces have been proved [5]-[8], [12]. The study of 2-metric spaces was initiated by S.Gähler [2] and some fixed point theorems in 2-metric spaces were proved in [1], [3], [4], [9]-[11], [13], [14].

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Some fixed point theorems in probabilistic 2-metric spaces [19] were proved in [1] and [15].

In this paper a fixed point theorem for multivalued $C-$ contractions in 2-Menger spaces will be proved. For singlevalued mappings the notion of a $C-$contraction in probabilistic metric spaces was introduced by Hicks [3].

2. Preliminaries

Let Δ^+ be the set of all the distribution functions F such that $F(0) = 0$ ($F : \mathbf{R} \to [0,1]$ is a nondecreasing and leftcontinuous mapping such that $\lim_{t\to\infty} F(t) = 0$).

Let $H \in \Delta^+$ be defined by :$H(t) = 1$, for $t > 0$ and $H(t) = 0$, for $t \leq 0$.

The notion of a Menger space is a generalization of the notion of a metric space [16]. In a similar way, the notion of a 2-Menger space is a generalization of the notion of a 2-metric space [19].

Let $S \neq \emptyset, \mathcal{F} : S \times S \times S \to \Delta^+$ and T be a $t-$ norm [16]. The triple $(S, \mathcal{F}, T)$ is a **2-Menger space** if the following conditions are satisfied:

1. For all $x, y \in S$ such that $x \neq y$ there exists $z \in S$ such that

$$F_{x,y,z} \neq H.$$

2. $F_{x,y,z} = H$ if at least two of $x, y, z \in S$ are equal.
3. For all $x, y, z \in S$, $F_{x,y,z} = F_{x,z,y} = F_{y,z,x}$.
4. For all $x, y, z, u \in S$ and for all $t_1, t_2, t_3 \geq 0$

$$F_{x,y,z}(t_1 + t_2 + t_3) \geq T^2(F_{x,y,u}(t_1), F_{x,u,z}(t_2), F_{u,y,z}(t_3)).$$

Remark. In this paper we shall use the following notation:

$$T^1(x,y) = T(x,y), T^2(x,y,z) = T(x, T^1(y,z)) \text{ and for every } n \geq 3,$$

$$T^n(x_1, x_2, ..., x_{n+1}) = T(x_1, T^{n-1}(x_2, x_3, ..., x_{n+1})).$$

A sequence $\{x_n\}_{n\in\mathbf{N}}$ from S converges to $x \in S$ if for every $a \in S$

$$F_{x,x_n,a} \to H$$

in the topology of the weak convergence in Δ^+. This means that for every $a \in S$, every $\epsilon > 0$ and every $\lambda \in (0,1)$ there exists $n(\epsilon, \lambda, a) \in \mathbf{N}$ such that

$$F_{x,x_n,a}(\epsilon) > 1 - \lambda, \text{ for all } n \geq n(\epsilon, \lambda, a).$$

A sequence $\{x_n\}_{n\in\mathbf{N}}$ from S is a Cauchy sequence if for every $a \in S, \epsilon > 0$ and $\lambda \in (0,1)$ there exists $n(\epsilon, \lambda, a) \in \mathbf{N}$ such that

$$F_{x_n,x_{n+p},a}(\epsilon) > 1 - \lambda$$

for every $n \geq n(\epsilon, \lambda, a)$ and every $p \in \mathbf{N}$.

A 2-Menger space is complete if every Cauchy sequence $\{x_n\}_{n\in\mathbf{N}}$ in S converges in S.

3. A fixed point theorem

In the Theorem we shall denote by $\mathcal{P}$ the set of all functions $\psi : \mathbf{R}^+ \to \mathbf{R}^+$ which are nondecreasing and such that for every $s > 0$ the series $\sum_{n=1}^{\infty} \psi^n(s)$ converges. By 2^M we shall denote the family of all nonempty subsets of $M \subset S$.

A $t-$norm T is of the $h-$type if the family of functions $\{T_n(x)\}_{n\in\mathbf{N}}$ is equicontinuous at the point $x = 1$, where $T_1(x) = T(x,x), T_n(x) = T(x, T_{n-1}(x))$, for every $n \geq 2$. In [5], such an example of a $t-$ norm T is given.

Theorem 1. *Let $(S, \mathcal{F}, T)$ be a complete 2-Menger space such that $\sup_{x<1} T(x,x) = 1, M$ be a nonempty and closed subset of $S, f : M \to 2^M$ a closed multivalued mapping, $\psi \in \mathcal{P}$ and the following conditions are satisfied:*

(i) $t-$ norm T is of the $h-$type.

(ii) For every $p, q \in S$ and every $x > 0$ the following implication holds:

$$F_{p,q,a}(x) > 1 - x, \textit{ for every } a \in S \Rightarrow \textit{ for every } u \in fp \textit{ there exists } v \in fq$$

$$\textit{such that } F_{u,v,a}(\psi(x)) > 1 - \psi(x), \textit{ for every } a \in S.$$

Then there exists $x \in M$ such that $x \in fx$.

Proof. Let $x_0 \in M$ and $x_1 \in fx_0$. Let $s > 1$. Then for every $a \in S$

(1) $$F_{x_1,x_0,a}(s) > 1 - s.$$

From (ii) and (1) follows that there exists $x_2 \in fx_1$ such that for every $a \in S$:

(2) $$F_{x_2,x_1,a}(\psi(s)) > 1 - \psi(s).$$

Similarly from (2) it follows that there exists $x_3 \in fx_2$ such that for every $a \in S$

$$F_{x_3,x_2,a}(\psi^2(s)) > 1 - \psi^2(s).$$

Hence, in the same way, we can obtain $x_n \in M, n \geq 3$ such that $x_n \in fx_{n-1}$ and that for every $a \in S$

(3) $$F_{x_n,x_{n-1},a}(\psi^{n-1}(s)) > 1 - \psi^{n-1}(s).$$

Since $\lim_{n\to\infty} \psi^n(s) = 0$, (3) implies that for every $a \in S$ and every $\epsilon > 0$

(4) $$\lim_{n\to\infty} F_{x_n,x_{n+1},a}(\epsilon) = 1.$$

Suppose that T is of the $h-$type. We shall prove that the sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence i.e. that for every $a \in S, \epsilon > 0$ and $\lambda \in (0,1)$ there exists $n_0(\epsilon, \lambda) \in \mathbf{N}$ such that

$$F_{x_{n+p},x_n,a}(\epsilon) > 1 - \lambda$$

for every $n \geq n_0(\epsilon, \lambda)$ and every $p \in \mathbf{N}$. Let $n'(\epsilon, s) \in \mathbf{N}$ be such that

$$2 \sum_{i \geq n'(\epsilon,s)} \psi^i(s) < \epsilon.$$

Then for every $n \geq n'(\epsilon, s)$ we have

$$F_{x_n,x_{n+p},a}(\epsilon) \geq T^2(F_{x_n,x_{n+1},x_{n+p}}(\psi^n(s)),$$

$$F_{x_n,x_{n+1},a}(\psi^n(s)), F_{x_{n+1},x_{n+p},a}(2 \sum_{i \geq n+1}^{n+p-2} \psi^i(s)))$$

$$\geq T^{2p-3}(F_{x_n,x_{n+1},x_{n+p}}(\psi^n(s)), F_{x_n,x_{n+1},a}(\psi^n(s)),$$

$$F_{x_{n+1},x_{n+2},x_{n+p}}(\psi^{n+1}(s)), F_{x_{n+1},x_{n+2},a}(\psi^{n+1}(s)), \ldots$$
$$F_{x_{n+p-2},x_{n+p-1},x_{n+p}}(\psi^{n+p-2}(s)), F_{x_{n+p-2},x_{n+p-1},a}(\psi^{n+p-2}(s))).$$

If $n \geq \max\{n(s), n'(\epsilon, s)\}$, where $\psi^n(s) < 1$, for every $n \geq n(s)$, then we have that

$$F_{x_n,x_{n+p},a}(\epsilon) \geq T^{2p-3}(1-\psi^n(s), 1-\psi^n(s), 1-\psi^{n+1}(s), 1-\psi^{n+1}(s), \ldots$$
$$1-\psi^{n+p-2}(s), 1-\psi^{n+p-2}(s)).$$

Since ψ is a nondecreasing function it follows that

$$F_{x_n,x_{n+p},a}(\epsilon) \geq T_{2p-3}(1-\psi^n(s)). \tag{5}$$

Using $\lim_{n\to\infty} \psi^n(s) = 0$ and the fact that T is of the $h-$type from (5) we conclude that $\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence.

Since S is complete there exists $\lim_{n\to\infty} x_n = x$. Using the relation $x_n \in fx_{n-1}$, for every $n \geq 1$ and the assumption that f is closed, we conclude that x is a fixed point of the mapping f.

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