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ON OPERATION-PARACOMPACT SPACES AND PRODUCTS

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Abstract

S. Kasahara [2] generalized the notion of compactness with the help of an operation of a topology into the power set of the space. In this paper we generalize the notion of paracompactness by using the operation and unify several product theorems for compact spaces and paracompact spaces.

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1. Definitions

Throughout this paper $(X, \mathcal{G})$ and $(Y, \mathcal{F})$ will represent topological spaces on which no separation axioms are assumed. Also, $(X \times Y, \mathcal{P})$ denotes the product space of two topological spaces $(X, \mathcal{G})$ and $(Y, \mathcal{F})$.

Definition 1. *([2]) A map γ from $\mathcal{G}$ to the power set $P(X)$ of X is said to be an operation on $(X, \mathcal{G})$ if $G \subset G^\gamma$ for each element G of $\mathcal{G}$, where G^γ denotes the value of γ at G.*

Definition 2. *([2]) Let γ be an operation on a topological space $(X, \mathcal{G})$. A subset K of X is said to be γ-compact if for every $\mathcal{G}$-open cover $\mathcal{C}$ of K there exists a finite subfamily $\{G_i;\ i = 1, \ldots, n\}$ of $\mathcal{C}$ such that the family $\{G_i^\gamma;\ i = 1, \ldots, n\}$ covers K.*

Definition 3. *Let γ be an operation on a topological space $(X, \mathcal{G})$. A subset K of X is said to be γ-paracompact (resp. γ-strongly paracompact, γ-weakly paracompact) if for every $\mathcal{G}$-open cover $\mathcal{C}$ there exists an $\mathcal{G}$-open family $\mathcal{B}$ such that*

(i) $\mathcal{B}$ is locally finite (resp. star finite, point finite).

(ii) $\mathcal{B}$ refines $\mathcal{C}$, and

(iii) $K \subset \bigcup\{B^\gamma;\ B \in \mathcal{B}\}$.

The aim of this paper is to present a common principle in proofs of several product theorems for compact spaces and paracompact spaces. The principle is stated in terms of operation mentioned above. Moreover, we introduce new classes of topological spaces and apply the principle to the product theorem for them.

Throughout the present paper, let γ (resp. η, ρ) be an operation on a topological space $(X, \mathcal{G})$ (resp. $(Y, \mathcal{F})$, $(X \times Y, \mathcal{P})$).

2. γ-compact spaces and products

In this section we study the following operator approaches of product theorems for compact spaces.

Theorem 1. *Suppose the operations γ, η and ρ satisfy the condition*

$$U^\gamma \times V^\eta \subset G^\rho \text{ for } U(\neq \emptyset) \in \mathcal{G}, V(\neq \emptyset) \in \mathcal{F} \text{ and } G \in \mathcal{P} \text{ with } U \times V \subset G. \tag{1}$$

Then $(X \times Y, \mathcal{P})$ is ρ-compact if $(X, \mathcal{G})$ and $(Y, \mathcal{F})$ are γ-compact and η-compact, respectively.

Proof. Let $\mathcal{C} = \{G_\lambda;\ \lambda \in \Lambda\}$ be a $\mathcal{P}$-open cover of $X \times Y$. For each point $(x, y) \in (X, Y)$ there exist $\lambda(x, y) \in \Lambda$, $U(x, y) \in \mathcal{G}$ and $V(x, y) \in \mathcal{F}$ such that $(x, y) \in U(x, y) \times V(x, y) \subset G_{\lambda(x,y)}$. For a fixed point $y \in Y$ there exists a finite number of points $x_1(y), \ldots, x_{n(y)}(y)$ of X such that

$$X = \bigcup\{U(x_i(y), y)^\gamma;\ i = 1, \ldots, n(y)\}.$$

Now, put

$$W_y = \bigcap\{V(x_i(y), y);\ i = 1, \ldots, n(y)\},$$

then there exist a finite number of points $y_1, \ldots, y_m$ of Y, such that $X \times Y$ is covered by the family

$$\{U(x_i(y_j), y_j)^\gamma \times W_{y_j}^\eta;\ j = 1, \ldots, m,\ i = 1, \ldots, n(y_j)\}.$$

For each $\mathcal{P}$-open set $U(x_i(y_i), y_i) \times W_{y_j}$ there exists $\mathcal{P}$-open set $G_{\lambda(x_i(y_j),y_j)} \in \mathcal{C}$ such that $U(x_i(y_j), y_j)^\gamma \times W_{y_j}^\eta \subset G_{\lambda(x_i(y_j),y_j)}^\rho$ by using the condition (1). Hence,

$$X \times Y = \bigcup\{G_{\lambda(x_i(y_j),y_j)}^\rho;\ j = 1, \ldots, m,\ i = 1, \ldots, n(y_j)\}.$$

The proof is complete.

Theorem 2. *Suppose the operations γ, η and ρ satisfy the condition*

(2) $$(U \times V)^\rho \subset U^\gamma \times V^\eta \text{ for } U(\neq \emptyset) \in \mathcal{G},\ V(\neq \emptyset) \in \mathcal{F}.$$

If $(X \times Y, \mathcal{P})$ is ρ-compact, then $(X, \mathcal{G})$ and $(Y, \mathcal{F})$ are γ-compact and η-compact, respectively.

Proof. Let $\{U_\lambda;\ \lambda \in \Lambda\}$ and $\{V_\mu;\ \mu \in M\}$ be an $\mathcal{G}$-open cover of X and a $\mathcal{F}$-open cover of Y, respectively. Then the family $\mathcal{C} = \{U_\lambda \times V_\mu;\ \lambda \in \Lambda,\ \mu \in M\}$ is a $\mathcal{P}$-open cover of $X \times Y$. Then there exists a finite subfamily $\{U_{\lambda_i} \times V_{\mu_i};\ i = 1, \ldots, n\}$ of $\mathcal{C}$ such that $X \times Y = \bigcup\{(U_{\lambda_i} \times V_{\mu_i})^\rho;\ i = 1, \ldots, n\}$. It follows from the condition (2) that $X \times Y = \bigcup\{U_{\lambda_i}^\gamma \times V_{\mu_i}^\eta;\ i = 1, \ldots, n\}$. Therefore, we have $X \subset \bigcup\{U_{\lambda_i}^\gamma;\ i = 1, \ldots, n\}$ and $Y \subset \bigcup\{V_{\mu_i}^\gamma;\ i = 1, \ldots, n\}$. The proof is complete.

3. γ-paracompact spaces and products

In this section we study a sufficient condition for the product space $(X \times Y, \mathcal{P})$ to be ρ-paracompact, ρ-strongly paracompact and ρ-weakly paracompact, respectively. We obtain the following operator approach of product theorems for paracompact spaces.

Theorem 3. *Suppose the operations γ, η and ρ satisfy the condition*

$$U^{\gamma} \times V^{\eta} \subset (U \times V)^{\rho} \text{ for } U(\neq \emptyset) \in \mathcal{G} \text{ and } V(\neq \emptyset) \in \mathcal{F}. \tag{3}$$

Then $(X \times Y, \mathcal{P})$ is ρ-paracompact (resp. ρ-strongly paracompact, ρ-weakly paracompact), if $(X, \mathcal{G})$ is γ-paracompact (resp. γ-strongly paracompact, γ-weakly paracompact) and $(Y, \mathcal{F})$ is η-compact.

Proof. Let $\mathcal{C}$ be a $\mathcal{P}$-open cover of $X \times Y$. Then for each point $(x, y) \in X \times Y$ there exist $C(x, y) \in \mathcal{C}$, $U(x, y) \in \mathcal{G}$ and $V(x, y) \in \mathcal{F}$ such that $(x, y) \in U(x, y) \times V(x, y) \subset C(x, y)$. For each fixed point $x \in X$ there exists a finite number of points $y_1(x), \ldots, y_{n(x)}(x) \in Y$ such that $Y = \bigcup\{V(x, y_i(x))^{\eta};\ i = 1, \ldots, n(x)\}$. Here we put $U_x = \bigcap\{U(x, y_i(x));\ i = 1, \ldots, n(x)\}$. As X is γ-paracompact (resp. γ-strongly paracompact, γ-weakly paracompact), we have an $\mathcal{G}$-open family $\mathcal{D} = \{W_{\mu};\ \mu \in M\}$ such that

(i) $\mathcal{D}$ is locally finite (resp. star finite, point finite),

(ii) $\mathcal{D}$ refines $\{U_x;\ x \in X\}$, and

(iii) $\{W_{\mu}^{\gamma};\ \mu \in M\}$ covers X.

For each $\mu \in M$ we take $x_{\mu} \in X$ such that $W_{\mu} \subset U_{x_{\mu}}$. Let

$$\mathcal{B} = \{W_{\mu} \times V(x_{\mu}, y_i(x_{\mu}));\ \mu \in M,\ i = 1, \ldots, n(x_{\mu})\}.$$

Then $\mathcal{B}$ is a locally finite (resp. star finite, point finite) $\mathcal{P}$-open family, and it refines $\mathcal{C}$. By using a condition (3) we can show that $\{B^{\rho};\ B \in \mathcal{B}\}$ covers $X \times Y$. This completes the proof.

4. Applications

Many authors have introduced and investigated many classes of compact spaces and paracompact spaces, e. g., almost compact spaces [10], nearly compact spaces [8], CO-compact spaces [4], almost CO-compact spaces [4], nearly paracompact spaces [6], almost paracompact spaces [7], weakly paracompact spaces, strongly paracompact spaces, nearly strongly paracompact spaces [3] and almost strongly paracompact spaces [3].

In this section, at first, we see that the notion of γ-paracompact (resp. γ-compact) spaces unifies the notions of paracompact (resp. compact) spaces mentioned above.

1. *If γ is the identity operation (resp. closure operation, interior-closure operation) in $(X, \mathcal{G})$, i. e., $G^\gamma = G$ (resp. $G^\gamma = \mathcal{G} - Cl(G)$, $G^\gamma = \mathcal{G} - Int(\mathcal{G} - Cl(G))$) ([2]), then the γ-compactness coincides with the compactness (resp. almost compactness, nearly compactness).*

2. *Let γ be an operation on $(X, \mathcal{G})$ defined by $G^\gamma = G$ (resp. $G^\gamma = \mathcal{G} - Cl(G)$) if G is CO-open ([4]) and $G^\gamma = X$ if G is not CO-open. Then γ-compactness coincides with the CO-compactness (resp. almost CO - compactness).*

3. *If γ is the identity operation (resp. closure operation, interior closure operation) on $(X, \mathcal{G})$, then the γ-paracompactness coincides with the paracompactness (resp. almost paracompactness, nearly paracompactness).*

4. *If γ is the identity operation (resp. closure operation, interior-closure operation) on $(X, \mathcal{G})$, then the γ-strongly paracompactness coincides with the strongly paracompactness (resp. almost strongly paracompactness, nearly strongly paracompactness [3], Def. 1.1 and Th. 1.5).*

5. *If γ is the identity operation on $(X, \mathcal{G})$, then the γ-weakly paracompactness coincides with the weakly paracompactness.*

To state applications we shall introduce new classes of topological spaces.

Definition 4. *A topological space $(X, \mathcal{G})$ is almost weakly paracompact (resp. nearly weakly paracompact) if $(X, \mathcal{G})$ is γ-weakly paracompact for γ to be the closure operation (resp. interior-closure operation) on $(X, \mathcal{G})$.*

By using Theorem 1, Theorem 2, Theorem 3 and statements 1, 2, 3, 4, 5 we have several known product theorems. In Theorem 3 let γ, η and ρ be interior-closure operators. We have the following

Theorem 4. *(I. Kovačević [3], Th. 4.1) The product of a nearly strongly paracompact space and a nearly compact space is nearly strongly paracompact.*

We also have the following theorem for the product of new classes of the topological spaces defined by Definition 4.

Theorem 5. *The product of a nearly (resp. almost) weakly paracompact space and a nearly (resp. almost) compact space is nearly (resp. almost) weakly paracompact.*

Proof. This follows from Theorem 3 for γ, η and ρ to be the interior-closure operation (resp. closure operation).

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