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ON COMPATIBLE MAPPINGS IN FIXED POINT THEORY

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Abstract

In [5], the concept of compatible mappings was introduced as a generalization of commuting mappings. The utility of compatibility in the context of fixed point theory was demonstrated in in many papers. In this paper, we extend the result of M. Imdad and A. Ahmad [4] in metric spaces, Y. J. Cho, K. S. Park, T. Mumtaz, M. S. Khan [1] in 2–metric spaces and M. Stojaković [8] in Menger spaces.

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1. Introduction

Definition 1. *Let f and g be a mappings from a metric space (X, d) into itself. Then $\{f, g\}$ are said to be* compatible *if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

Mappings which commute are clearly compatible, but converse is false. S. Sessa [7] generalized commuting mappings by calling mappings f and g from a metric space (X,d) into itself a weakly commuting pair if $d(fgx,gfx) \leq d(gx,fx)$ for all $x \in X$. Any weakly commuting pair are obviously compatible, but the converse is false. See [5] for examples of the compatible pairs which are not weakly commutative and hence not commuting pairs.

The following Lemma will be usefull later.

Lemma. *Let f and g be compatible mappings from metric space (X,d) into itself. If f is continuous and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ then $\lim_{n\to\infty} gfx_n = fz$.*

2. Result in metric space

Theorem 1. *Let $\{S,I\}$ and $\{T,J\}$ be compatible pairs of mappings of complete metric space (X,d) into itself such that*

a) $T(X) \subset I(X), \qquad S(X) \subset J(X);$

b) For all x, y in X either

$$(1) \quad d(Sx,Ty) \leq \alpha \frac{d(Ix,Sx)d(Ix,Ty) + d(Jy,Ty)d(Jy,Sx)}{d(Ix,Ty) + d(Jy,Sx)} + \beta d(Ix,Jy)$$

if $d(Ix,Ty) + d(Jy,Sx) \neq 0$, *where* $\alpha, \beta > 0, \; \alpha + \beta < 1$, *or*

$$(1') \qquad d(Sx,Ty) = 0 \qquad \text{if} \qquad d(Ix,Ty) + d(Jy,Sx) = 0$$

If one of S,T,I or J is continuous, then S,T,I and J have unique fixed point z. Further, z is the unique common fixed point of S and I and T and J.

Proof. Let x_0 be an arbitrary point of X. Since $S(X) \subset J(X)$ we can find a point x_1 in X such that $Sx_0 = Jx_1$. Also, since $T(X) \subset I(X)$ we can choose a point x_2 with $Tx_1 = Ix_2$. In general for the point x_{2n} we can pick up a point x_{2n+1} such that $Sx_{2n} = Jx_{2n+1}$ and then a point x_{2n+2} with $Tx_{2n+1} = Ix_{2n+2}$ for $n = 0, 1, 2, \ldots$.

Let us put $u_{2n} = d(Sx_{2n}, Tx_{2n+1})$ and $u_{2n+1} = d(Tx_{2n+1}, Sx_{2n+2})$.

Now, we distinguish two cases:

(i) Suppose $u_{2n} \neq 0, u_{2n+1} \neq 0$ for $n = 0, 1, 2, \ldots$.

Then, on using inequality (1), we have

$$u_{2n+1} \leq (\alpha + \beta) u_{2n} \leq \ldots \leq (\alpha + \beta)^{2n+1} \cdot u_0, \text{ for } n = 0, 1, 2, \ldots. \tag{2}$$

It follows that the sequence

$$\{Sx_0, Tx_1, Sx_2, \ldots, Tx_{2n-1}, Sx_{2n}, Tx_{2n+1}, \ldots\}$$

is Cauchy sequence in the complete metric space (X, d) and so gets a limit z in X. Hence the sequences $\{Sx_{2n}\} = \{Jx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Ix_{2n}\}$ as it's subsequences also converge to the same point z.

Let us now suppose that I is continuous so that the sequences $\{I^2 x_{2n}\}$ and $\{ISx_{2n}\}$ converge to the same point Iz. Since S and I are compatible by continuity of I and Lemma we have the sequence $\{SIx_{2n}\}$ also converges to the point Iz.

As in [4] one can see that

$$\begin{aligned} d(SIx_{2n}, Tx_{2n+1}) \quad \leq \quad & \alpha \left(\frac{d(I^2 x_{2n}, SIx_{2n}) d(I^2 x_{2n}, Tx_{2n+1})}{d(I^2 x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, SIx_{2n})} + \right. \\ & \left. + \frac{d(Jx_{2n+1}, Tx_{2n+1}) d(Jx_{2n+1}, SIx_{2n})}{d(I^2 x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, SIx_{2n})} \right) + \\ & + \beta d(I^2 x_{2n}, Jx_{2n+1}) \end{aligned}$$

which on letting $n \to \infty$ reduced to

$$d(Iz, z) \leq \beta d(Iz, z),$$

giving $Iz = z$.

Similarly $d(Sz, z) = 0$ and hence $Sz = z$.

Since $Sz = z$ and $S(X) \subset J(X)$ there always exists a point z^* such that $Jz^* = z$. Thus

$$\begin{aligned} d(z, Tz^*) = d(Sz, Tz^*) \quad \leq \quad & \alpha \frac{d(Iz, Tz) d(Iz, Tz^*) + d(Jz^*, Tz^*) d(Jz^*, Sz)}{d(Iz, Tz^*) + d(Jz^*, Sz)} \\ & + \beta d(Iz, Jz^*) \end{aligned}$$

giving thereby $Tz^* = z$.

Using compatibility, $Tz^* = Jz^*$ implies $d(JTz^*, TJz^*) = 0$ and hence $Tz = TJz^* = JTz^* = Jz$. At the end we have that

$$\begin{aligned} d(z,Tz) = d(Sz,Tz) &\leq \alpha \frac{d(Iz,Sz)d(Iz,Tz) + d(Jz,Tz)d(Jz,Sz)}{d(Iz,Tz) + d(Jz,Sz)} + \\ &\quad + \beta d(Iz,Jz) \\ &= \beta d(z,Tz) \end{aligned}$$

which implies that $z = Tz = Jz$.

Thus, we have proved that z is a common fixed point of S, T, I and J.

Now suppose that S is continuous, so that the sequences $\{S^2 x_{2n}\}$ and $\{SIx_{2n}\}$ converge to the point Sz. Since S and I are compatible, it follows similarly that sequence $\{ISx_{2n}\}$ also converge to Sz and that $Sz = z$.

As $S(X) \subset J(X)$ and $Sz = z$, we can find a point z^* in X such that $Jz^* = z$, and show that $Tz^* = z$. Since T and J are compatibility it again follows as above that $Tz = Jz$. Further

$$\begin{aligned} d(Sx_{2n},Tz) &\leq \alpha \left(\frac{d(Ix_{2n},Sx_{2n})d(Ix_{2n},Tz)}{d(Ix_{2n},Tz) + d(Jz,Sx_{2n})} + \right. \\ &\quad \left. + \frac{d(Jz,Tz)d(Jz,Sx_{2n})}{d(Ix_{2n},Tz) + d(Jz,Sx_{2n})} \right) \\ &\quad + \beta d(Ix_{2n},Jz) \end{aligned}$$

which on making $u \to \infty$ gives $z = Tz$.

Thus, the point z is in the range of T and since $T(X) \subset I(X)$ there always exists a point $\tilde{z}$ in X such that $I\tilde{z} = z$

Thus, on (1),

$$\begin{aligned} d(S\tilde{z},z) = d(S\tilde{z},Tz) &\leq \alpha \left(\frac{d(I\tilde{z},S\tilde{z})d(I\tilde{z},Tz)}{d(I\tilde{z},Tz) + d(Jz,S\tilde{z})} + \right. \\ &\quad \left. + \frac{d(Jz,Tz)d(Jz,S\tilde{z})}{d(I\tilde{z},Tz) + d(Jz,S\tilde{z})} \right) \\ &\quad + \beta d(I\tilde{z},Jz) = 0 \end{aligned}$$

so $S\tilde{z} = z$.

Again, since S and I are compatible, $S\tilde{z} = I\tilde{z}$ we have that $d(SI\tilde{z}, IS\tilde{z}) = 0$ so

$$Sz = SI\tilde{z} = IS\tilde{z} = Iz$$

Thus, we have proved again that z is a common fixed point of S, T, I and J.

If the mapping T or J is continuous instead of S or I, then the proof that z is a common fixed point of S, T, I and J is similar.

One can show that z is unique common fixed point for S and I and T and J.

(ii) If $u_{2n} = 0$, for some n, then inequality (2) gives $u_{2n+1} = 0$ which implies that

$$Sx_{2n} = Jx_{2n+1} = Tx_{2n+1} = Ix_{2n+2} = Sx_{2n+2} = \cdots = z.$$

As in [4] one can argue that z is a unique fixed point of S, T, I and J.

Remark. By choosing α, β, I, J, S and T suitably, we can derive a multitude of fixed point theorems which generalized well known results for weakly commuting mappings.

3. Results in 2–metric spaces

At first let us recall the notion of 2–metric spaces.

Definition 2. *A 2–metric space is a nonempty set X with a real-valued function d on $X \times X \times X$ satisfying the following conditions:*

1) For two distinct points x, y in X there is a point z in X such that $d(x,y,z) \neq 0$;

2) $d(x,y,z) = 0$ if at least two of x, y, z are equal;

3) $d(x,y,z) = d(x,z,y) = d(y,z,x)$ for all x, y, z in X;

4) $d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z)$ for all x, y, z, u in X.

Function d is called a 2–metric for the space X and (X,d) is called a 2–metric space.

Remark. It has been shown that although d is a continuous function of any one of its three arguments it need not be continuous in two arguments but if it is continuous in two arguments then it is continuous in all three arguments.

Number of mathematicians have studied the aspects of fixed point theory in the setting of 2–metric spaces. They have been motivatied by various concepts known for the metric spaces and have thus introduced analogues of various concepts in the frame work of 2–metric spaces. Let us recall definition of asymptotically regular sequence and define the notion of compatible mappings in 2-metric spaces.

Definition 3. *Let (X,d) be a 2–metric space, and S and T be mappings from X into itself. Then a sequence $\{x_n\}$ in X is said to be asymptotically $\{S,T\}$–regular if*

$$\lim_{n\to\infty} d(Tx_n, Sx_n, a) = 0$$

for all a in X.

In 2–metric space the notion of compatibility has the following form.

Definition 4. *Let f and g be mappings from 2–metric space X into itself. Then $\{f,g\}$ are said to be compatible if for every $a \in X$*

$$\lim_{n\to\infty} d(fgx_n, gfx_n, a) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X.

Theorem 2. *Let (X,d) be a complete 2–metric space, d continuous and A, S and T be mappings from X into itself such that*

(1) S and T are sequentialy continuous;

(2) $\{A,S\}$ and $\{A,T\}$ are compatible pairs;

(3) There exists an asymptotically $\{A,S\}$ and $\{A,T\}$–regular sequence;

(4) $$d(Ax,Ay,a) \le a_1 d(Sx,Ax,a) + a_2 d(Tx,Ax,a) + a_3 d(Sy,Ay,a) + a_4 d(Ty,Ay,a) + a_5 d(Sx,Ay,a) + a_6 d(Tx,Ay,a)$$

$$+a_7 d(Sy, Ax, a) + a_8 d(Ty, Ax, a) + a_9 d(Sx, Ty, a)$$
$$+a_{10} d(Sy, Tx, a)$$

for all x, y, a in X, where a_i, $i = 1, 2, \ldots, 10$, are non-negative real numbers such that

$$\max\{a_5 + a_6 + \cdots + a_{10},\ \ a_2 + a_3 + a_5 + a_8 + a_9 + a_{10},$$

$$a_3 + a_4 + a_5 + a_6,\ \ a_1 + a_2 + a_7 + a_8\} < 1.$$

Then A, S and T have a unique common fixed point in X.

Proof. Let $\{x_n\}$ be an asymptotically $\{A, S\}$ and $\{A, T\}$–regular sequence.

Then by (4)

$$d(Ax_n, Ax_m, a) \leq a_1 d(Sx_n, Ax_n, a) + a_2 d(Tx_n, Ax_n, a) +$$

$$+a_3 d(Sx_m, Ax_m, a) + a_4 d(Tx_m, Ax_m, a) + a_5 d(Sx_n, Ax_m, a) +$$

$$+a_6 d(Tx_n, Ax_m, a) + a_7 d(Sx_m, Ax_n, a) + a_8 d(Tx_m, Ax_n, a) +$$

$$+a_9 d(Sx_n, Tx_m, a) + a_{10} d(Sx_m, Tx_n, a)$$

for all a in X and hence, by condition 4) of 2–metric

$$(1 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}) d(Ax_n, Ax_m, a) \leq$$

$$\leq (a_1 + a_5 + a_9) d(Sx_n, Ax_n, a) + (a_2 + a_6 + a_{10}) d(Tx_n, Ax_n, a) +$$

$$+(a_3 + a_7 + a_{10}) d(Sx_m, Ax_m, a) + (a_4 + a_8 + a_9) d(Tx_m, Ax_m, a) +$$

$$+(a_8 + a_9) d(Tx_m, Ax_n, Ax_m) + (a_6 + a_{10}) d(Ax_m, Ax_n, Tx_n) +$$

$$+a_5 d(Sx_n, Ax_m, Ax_n) + a_7 d(Sx_m, Ax_n, Ax_m) + a_9 d(Sx_n, Tx_m, Ax_n) +$$

$$+a_{10} d(Sx_m, Tx_n, Ax_m)$$

for all a in X.

Since $\{x_n\}$ is an asymptotically $\{A, S\}$ and $\{A, T\}$–regular sequence, as $m, n \to \infty$, we have

$$(1 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10}) d(Ax_n, Ax_m, a) \to 0$$

for all a in X.

Therefore, $\{Ax_n\}$ is a Cauchy sequence in X. Since (X, d) is complete 2–metric space, $\{Ax_n\}$ has the limit z in X. It means that $\lim_{n\to\infty} d(Ax_n, z, a) = 0$ for all a in X.

Since

$$\begin{aligned} d(Sx_n, z, a) \quad &\leq \quad d(Sx_n, z, Ax_n) + d(Sx_n, Ax_n, a) + \\ &\quad +d(Ax_n, z, a) \to 0, \qquad n \to \infty, \end{aligned}$$

$Sx_n \to z$ as $n \to \infty$. Similarly, we have $Tx_n \to z$, $n \to \infty$. Maps S and T are sequentially continuous, so it follows that

$$SAx_n \to Sz,\ S^2x_n \to Sz,\ STx_n \to Sz,\ TAx_n \to Tz,\ T^2x_n \to Tz,$$
$$TSx_n \to Tz, \quad \text{as} \quad n \to \infty.$$

Since

$$\begin{aligned} d(ATx_n, Tz, a) \quad &\leq \quad d(ATx_n, Tz, TAx_n) + d(ATx_n, TAx_n, a) \\ &\quad +d(TAx_n, Tz, a) \end{aligned}$$

and $\{A, T\}$ are compatible one can see that

$$\lim_{n\to\infty} ATx_n = Tz$$

(of course you can use Lemma, too).
Similarly, we have also $ASx_n \to Sz$ as $n \to \infty$.

One can prove, just as in [1], that $Sz = Tz = Az$.

Let us prove that $Az = A^2z$. At first notice that

$$\lim_{n\to\infty} d(A(Sx_n), S(Sx_n), a) = 0 \qquad \text{and}$$

$$\lim_{n\to\infty} d(A(Sx_n), T(Sx_n), a) = 0 \qquad \text{for each} \qquad a \in X.$$

so $\{Sx_n\}$ is asymptotically $\{A, S\}$ and $\{A, T\}$–regular. Now, if we repeat the above procedure for sequence $\{Sx_n\}$, using that $\lim_{n\to\infty} A(Sx_n) = Az$, we have that

$$A(Az) = S(Az) = T(Az).$$

Now by 4), for all a in X,

$$\begin{aligned} d(Az, A^2z, a) &\le a_1 d(Sz, Az, a) + a_2 d(Tz, Az, a) + a_3 d(SAz, A^2z, a) + \\ &+ a_4 d(TAz, A^2z, a) + a_5 d(Sz, A^2z, a) + \\ &+ a_6 d(Tz, A^2z, a) + a_7 d(SAz, Az, a) + a_8 d(TAz, Az, a) \\ &+ a_9 d(Sz, TAz, a) + a_{10}(SAz, Tz, a) \\ &= (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) d(Az, A^2z, a) \end{aligned}$$

Since $a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} < 1$, $d(Az, A^2z, a) = 0$ that is $Az = A^2z$. Putting $p = Az$ we have that

$$p = Ap = Sp = Tp.$$

Thus, p is a common fixed point of A, S and T. For uniqueness see [1].

Remark. A is sequentially continuous at common fixed point of A, S and T in this case too.

4. Result in Menger space

A Menger space is a space in which the concept of distance is considered to be probabilistic, rather then deterministic. For a detailed discussion of Menger spaces and their applications we refer to Schweizer and Sklar [6]. The theory of Menger spaces is of fundamental importance in probabilistic functional analysis. Recently, some fixed point theorems for mappings in Menger spaces have been proved by several authors: G. Bocsan, A. F. Bharucha-Raid, S. S. Chang, Gh. Constantin, O. Hadžić and others (see [3]).

Let $\mathbf{R}$ denote the reals and $\mathbf{R}^+ = \{x \in \mathbf{R} : x \ge 0\}$. A mapping $F : \mathbf{R} \to \mathbf{R}^+$ is called a distribution function if it is non-decreasing, left continuous with $\inf F = 0$ and $\sup F = 1$. We will denote by $\triangle$ the set of all distribution functions. A commutative, associative and non-decreasing mapping $t : [0,1] \times [0,1] \to [0,1]$ is a $T-$norm if and only if $t(a, 1) = a$ for all $a \in [0,1]$ and $t(0,0) = 0$.

Definition 5. *A Menger space is a triple $(X, \mathcal{F}, t)$ where X is a set, $\mathcal{F}$ is a mapping from $X \times X$ into $\triangle$ and t is a $T-$norm. We shall denote the distribution function $\mathcal{F}(x,y)$ by $F_{x,y}$ and $F_{x,y}(\varepsilon)$ will represent the value of $F_{x,y}$ at $\varepsilon \in \mathbf{R}$. The function $F_{x,y}$, $x, y \in X$, are assumed to satisfy the following conditions:*

1. $F_{x,y}(\varepsilon) = 1$ *for* $\varepsilon > 0$ *if and only if* $x = y$.

2. $F_{x,y}(0) = 0$, *for all* $x, y \in X$.

3. $F_{x,y} = F_{y,x}$, *for all* $x, y \in X$.

4. $F_{x,y}(\varepsilon + \delta) \geq t(F_{x,z}(\varepsilon), F_{z,y}(\delta))$, *for all* $x, y, z \in X$.

Throughout this paper H will denote the specific distribution function defined by

$$H(\varepsilon) = \begin{cases} 0 & \varepsilon \leq 0 \\ 1 & \varepsilon > 0. \end{cases}$$

The concept of neighbourhoods in Menger space was introduced by Schweizer and Sklar [6]. If $x \in X$, $\varepsilon > 0$ and $\lambda \in (0,1)$, then an $(\varepsilon, \lambda)-$neighbourhood of x, called $U_x(\varepsilon, \lambda)$, is defined by

$$U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}.$$

If t is continuous, then $(X, \mathcal{F}, t)$ is a Hausdorff space in the topology induced by the family $\{U_x(\varepsilon, \lambda) : x \in X, \varepsilon > 0, \lambda \in (0,1)\}$ of neighbourhoods.

Definition 6. *A set* $M \subseteq X$ *is called probabilistically bounded if and only if*

$$\sup_{\varepsilon > 0} \inf_{x,y \in M} F_{x,y}(\varepsilon) = 1.$$

Definition 7. *The pair* $\{f, g\}$, *where* $f : X \to X$ *and* $g : X \to X$, *is compatible if for every sequence* $\{x_n\} \subset X$ *such that* $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z \in X$ *the relation*

$$\lim_{n \to \infty} F_{fgx_n, gfx_n}(\varepsilon) = H(\varepsilon)$$

holds for all $\varepsilon \in \mathbf{R}$.

Theorem 3. *Let* $(X, \mathcal{F}, t)$ *be a complete Menger space with a continuous* $T-$*norm* t *and let* $h : X \to X$, $k : X \to X$, $f : X \to h(X)$ *and* $g : X \to k(X)$ *be continuous mappings such that* $\{f, k\}$ *and* $\{g, h\}$ *are compatible pairs. Further, suppose that for all* $x, y \in X$ *and for all* $\varepsilon > 0$ *the following inequality holds*

$$(*) \qquad F_{fx,gy}(\varepsilon) \geq F_{kx,hy}(\phi(\varepsilon)),$$

where $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ *is an increasing function such that* $\lim_{n\to\infty} \phi^n(t) = \infty$ *for all* $t > 0$. *If the sequence* $\{y_n\}_{n\in N}$ *formed by*

$$y_{2n-1} = gx_{2n-1} = kx_{2n},$$

$$y_{2n} = fx_{2n} = hx_{2n+1}, \quad n \in N$$

is probabilistically bounded for some $x_1 \in X$, *then there exists a unique common fixed point for the mappings* f, g, h *and* k.

Proof. Let $\{y_n\}$ be the sequence as was noted above. First, we shall show that $\{y_n\}_{n\in N}$ is a Cauchy sequence.

In order to prove that, we shall show that

$$\lim_{\substack{m\to\infty \\ p\to\infty}} F_{y_m,y_p}(\varepsilon) = H(\varepsilon), \qquad \text{for every} \qquad \varepsilon \in \mathbf{R}.$$

If $m = 2i$ and $p = 2j-1$ (let $j > i$) then we have

$$\begin{aligned} F_{y_{2i},y_{2j-1}}(\varepsilon) &= F_{fx_{2i},gx_{2j-1}}(\varepsilon) \geq F_{kx_{2i},hx_{2j-1}}(\phi(\varepsilon)) \\ &= F_{fx_{2j-2},gx_{2i-1}}(\phi(\varepsilon)) \geq F_{kx_{2j-2},hx_{2i-1}}(\phi^2(\varepsilon)) \\ &= F_{fx_{2i-2},gx_{2j-3}}(\phi^2(\varepsilon)) \geq F_{fx_0,gx_{2j-1-2i}}(\phi^{2i}(\varepsilon)) \\ &\geq \sup_{t<\phi^{2i}(\varepsilon)} \inf_{n,k\in N} F_{y_n,y_k}(t) = D_{\{y_n\}_{n=1}^\infty}(\phi^{2i}(\varepsilon)). \end{aligned}$$

Since $\{y_n\}_{n\in N}$ is probabilistically bounded, letting $i \to \infty$ and $j \to \infty$, we get

$$\lim_{i\to\infty} D_{\{y_n\}_{n=1}^\infty}(\phi^{2i}(\varepsilon)) = H(\varepsilon).$$

Repeating this procedure we can prove a similar result for $m = 2i-1$ and $p = 2j$.

If m and p are both even or both odd, we proceed as follows.

$$F_{y_{2i},y_{2j}}(\varepsilon) \geq t(F_{y_{2i},y_{2i+1}}(\frac{\varepsilon}{2}), F_{y_{2i+1},y_{2j}}(\frac{\varepsilon}{2})) \to t(H(\varepsilon), H(\varepsilon)) = H(\varepsilon),$$

$$F_{y_{2i-1},y_{2j-1}}(\varepsilon) \geq t(F_{y_{2i-1},y_{2i}}(\frac{\varepsilon}{2}), F_{y_{2i},y_{2j-1}}(\frac{\varepsilon}{2})) \to t(H(\varepsilon), H(\varepsilon)) \to H(\varepsilon)$$

if $i \to \infty$ and $j \to \infty$, for all $\varepsilon > 0$.

Thus, we have proved that $\{y_n\}_{n\in N}$ is a Cauchy sequence in X which means that there exists $y^* \in X$ such that $\lim\limits_{n\to\infty} y_n = y^*$.

To establish that $fy^* = gy^* = hy^* = ky^*$, we proceed as follows.

$$\begin{aligned} fy^* &= f \lim_{n\to\infty} kx_{2n} = \lim_{n\to\infty} fkx_{2n} = \lim_{n\to\infty} kfx_{2n} = \\ &= k \lim_{n\to\infty} fx_{2n} = ky^* \end{aligned}$$

$$\begin{aligned} gy^* &= g \lim_{n\to\infty} hx_{2n+1} = \lim_{n\to\infty} ghx_{2n+1} = \lim_{n\to\infty} hgx_{2n+1} \\ &= h \lim_{n\to\infty} gx_{2n+1} = hy^*. \end{aligned}$$

Since

$$F_{fy^*,gy^*}(\varepsilon) \geq F_{ky^*,hy^*}(\phi(\varepsilon)) = F_{fy^*,gy^*}(\phi(\varepsilon)) \geq \dots$$
$$\dots \geq F_{fy^*,gy^*}(\phi^n(\varepsilon)) \to H(\varepsilon) \qquad \text{for all} \qquad \varepsilon > 0.$$

we have proved that $fy^* = gy^* = ky^* = hy^*$.

The point fy^* is a fixed point for the mappings f, g, h, k. We shall show this for the mapping f; the proof for the mappings g, h, k is analogous.

From compatibility of $\{f, k\}$ we obtain that

$$kfy^* = kf \lim_{n\to\infty} y_n = \lim_{n\to\infty} kfy_n = \lim_{n\to\infty} fky_n = fk \lim_{n\to\infty} y_n = fky^* = ffy^*.$$

Further,

$$F_{ffy^*,fy^*}(\varepsilon) = F_{ffy^*,gy^*}(\varepsilon) \geq F_{kfy^*,hy^*}(\phi(\varepsilon)) = F_{ffy^*,fy^*}(\phi(\varepsilon)) \geq \dots$$
$$\dots \geq F_{ffy^*,fy^*}(\phi^n(\varepsilon)) \to H(\varepsilon) \qquad \text{for} \qquad n \to \infty,$$

for all $\varepsilon > 0$, which means that fy^* is a common fixed point for the mappings f, g, h and k.

If we suppose that there exists another common fixed point $z \in X$, we get

$$F_{fy^*,z}(\varepsilon) = F_{ffy^*,gz}(\varepsilon) \geq F_{kfy^*,hz}(\phi(\varepsilon)) =$$
$$F_{ffy^*,gz}(\phi(\varepsilon)) \geq \dots \geq F_{fy^*,z}(\phi^n(\varepsilon)) \to H(\varepsilon)$$

for $n \to \infty$ for all $\varepsilon > 0$, which means that fy^* is a unique common fixed point for the mappings f, g, h and k.

This completes the proof of Theorem 3.

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