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THE OPERATOR GREEN FUNCTION

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Abstract

We construct the numerical solution for a class of operator differential equations with the coefficients depending on one variable by using the Green function and the corresponding differential analogues in the field of Mikusiński operators $\mathcal{F}$ (see [1]). Also, we estimate the error of approximation.

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1. Introduction

In this paper we consider the following linear partial differential equation with variable coefficients

$$\frac{\partial^{2+p}u(\lambda,t)}{\partial t^p \partial \lambda^2} - \sum_{i=0}^{r} B_i(\lambda)\frac{\partial^i u(\lambda,t)}{\partial t^i} = f_1(\lambda,t), \tag{1}$$

with the appropriate conditions

$$\frac{\partial^{\mu+\nu}u(\lambda,0)}{\partial \lambda^\mu \partial t^\nu} = 0; \qquad \begin{array}{l} \mu = 0; \nu = 0,1,...,r-1, \\ \mu = 2; \nu = 0,1,...,p-1, \end{array} \tag{2}$$

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$$(3) \qquad x(0,t) = C(t), \qquad x(1,t) = D(t),$$

where $t \geq 0$, $\lambda \in [0,1]$, $p, r \in N$, $p \geq r$, $B_i(\lambda), i = 0, ..., r$, are real valued continuous functions of one variable, while $f_1(\lambda, t)$ is a continuous real valued function of two variables. $C(t)$ and $D(t)$ are continuous functions for $t \geq 0$.

The ring of continuous functions $\mathcal{C}$ on $[0, \infty)$ (or locally integrable functions $\mathcal{L}$ on $[0, \infty)$) with usual addition and multiplication given by the convolution

$$f(t) * g(t) = \int_0^t f(\tau) g(t-\tau) d\tau,$$

where f and g are from $\mathcal{C}$ (or from $\mathcal{L}$), has no divisors of zero, hence its quotient field can be defined. The elements of this field, called the field of Mikusiński operators $\mathcal{F}$, are of the form

$$\frac{f}{g},$$

where this division is observed in the sense of convolution. The most important operators are the integral operator l and its inverse operator, the differential operator s, while I is the identical operator. It holds

$$ls = I, \quad l^\alpha = \{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\}, \quad \alpha > 0$$

$$\{x^{(n)}(t)\} = s^n x - s^{n-1} x'(0) - \ldots - x^{(n-1)}(0) I.$$

Let us denote by $\mathcal{F}_c$ the special subset of $\mathcal{F}$ consisting of the operators representing continuous functions.

An operational function $g(\lambda)$ is continuous ([1] pp. 191) in a finite (open or closed) interval J if it can be represented as

$$g(\lambda) = q\{g_1(\lambda, t)\}$$

where $g_1(\lambda, t)$ is a continuous function (in the usual sense) in the domain $D = \{(\lambda, t), \lambda \in J, t \geq 0\}$, and q is some operator from $\mathcal{F}$.

An operational function $g(\lambda)$ has a continuous n-th derivative $g^{(n)}(\lambda)$ in an interval J if there exist an operator q_n and a parametric function

$\{g_n(\lambda, t)\}$ with a continuous partial derivative $\{\frac{\partial^n}{\partial \lambda^n} g_n(\lambda, t)\}$ in the domain D, such that

$$g(\lambda) = q_n\{g_n(\lambda, t)\}, \quad g^{(n)}(\lambda) = q_n\{\frac{\partial^n}{\partial \lambda^n} g_n(\lambda, t)\}.$$

In the field of Mikusiński operators, $\mathcal{F}$, the problem (1), (2) corresponds to the following problem

$$u''(\lambda) - Q(\lambda)u(\lambda) = f(\lambda), \tag{4}$$

$$u(0) = C, \quad u(1) = D, \tag{5}$$

where $Q(\lambda)$ and $f(\lambda)$ are continuous operator functions. Since $p \geq r$, the operator function $Q(\lambda)$ can be written as

$$Q(\lambda) = Q^1(\lambda)I + Q^c(\lambda), \tag{6}$$

where $Q^1(\lambda)$ is operator function representing continuous function of only one variable λ, while $Q^c(\lambda)$, represent continuous function of two variables λ, t. Further on, the operator function $f(\lambda)$ represent continuous function of two variables λ, t and the operators C and D are from $\mathcal{F}_c$.

Let us suppose that $Q(\lambda)$, and $f(\lambda)$ are operator functions which have continuous second derivatives and the solution $u(\lambda)$ is an operator function which has a continuous fourth derivative. (This means that the functions $Q\ (\lambda, t), f\ (\lambda, t)$ and $u(\lambda, t)$ corresponding to operator functions $Q(\lambda)$, and $f(\lambda)$ and $u(\lambda)$ have continuous derivatives of the mentioned order by λ, but not necessarily by t. From relation (6) it follows that the operator function $lQ(\lambda)$ represents the function continuous by t, too.

2. Operator Green function

In this section we shall construct the Green function in the field of Mikusiński operators.

Let us denote by $G(\lambda, r)$ the function from the field of Mikusiński operators, which is the solution of the problem

$$G''(\lambda, r) - Q(\lambda)G(\lambda, r) = \delta(\lambda - r)I, \tag{7}$$

$$G(0,r) = G(1,r) = 0, \tag{8}$$

where the operator function $Q(\lambda)$ is given by relations (6) and the numerical function and

$$\delta(\lambda - r) = \begin{cases} 0, & \lambda = r, \\ 1, & \lambda \neq r. \end{cases}$$

Similarly as in the classical case we have that the solution of the problem (4), (5) in that case can be written in the form

$$u(\lambda) = \int_0^1 G(\lambda,r)f(r)dr + G(\lambda,1)C - G_r(\lambda,0)D. \tag{9}$$

The operator functions $W^1(\lambda), W^2(\lambda)$ can be defined as the solutions of the corresponding homogeneous equation

$$L(u) \equiv u''(\lambda) - Q(\lambda)u(\lambda) = 0$$

satisfying the conditions

$$W^1(0) = 0, \quad (W^1)'(0) = I,$$

$$W^2(1) = 0, \quad (W^2)'(1) = -I.$$

From the previous relations and relation (6) it follows that the operator functions $W^1(\lambda), W^2(\lambda)$ can be written in the form

$$\begin{aligned} W^1(\lambda) &= w^1(\lambda)I + w_{1,c}(\lambda), \\ W^2(\lambda) &= w^2(\lambda)I + w_{2,c}(\lambda), \end{aligned} \tag{10}$$

where $w^1(\lambda), w^2(\lambda)$ are the operator functions representing continuous functions of only one variable λ, while $w_{1,c}(\lambda)$, and $w_{2,c}(\lambda)$ represent continuous functions of two variables λ, t.

The operator Green function can be constructed as

$$G(\lambda,r) = \begin{cases} \dfrac{W^2(r)W^1(\lambda)}{V^0} & 0 \leq \lambda \leq r \\[2ex] \dfrac{W^1(r)W^2(\lambda)}{V^0} & r \leq \lambda \leq 1, \end{cases} \tag{11}$$

where V_0 is and has the form

$$V(\lambda) = \begin{vmatrix} W^1(\lambda) & W^2(\lambda) \\ (W^1)'(\lambda) & (W^2)'(\lambda) \end{vmatrix} = V_0.$$

Similarly as in the classical case it can be shown that V_0 is a constant operator (not depending on λ), and can written as

$$V_0 = v_0 I + v_c,$$

where the operator v_0 is a numerical constant, and the operator v_c represents a continuous function.

Therefore from relation (10) it follows that in this case the operator Green function can be written as

$$G(\lambda, r) = g^1(\lambda, r)I + g_{1,c}(\lambda, r), \tag{12}$$

where $g^1(\lambda, r)$, is the operator function representing continuous functions of variables λ, r while $g_{1,c}(\lambda, r)$, represent continuous function of three variables λ, r, t.

If $Q(\lambda)$ is of the form (6), $f(\lambda)$ represents the continuous function of two variables λ, t and C and D are the operators from $\mathcal{F}_c$, then it follows that the exact solution of the problem (4), (5) $u(\lambda)$, given by relation (9), is the operator function which can be represented by continuous function of two variables λ, t.

This means that we can use the operator Green function to obtain the solution in the field of Mikusiński operators. This solution is very useful because it represent the continuous function.

3. Discrete Green operator

In this chapter we shall construct the Green operator for the discrete analogues for the problem (4), (5). As is usual in numerical analysis, let us take the following notations:

$$h = \frac{1}{n+1}, \quad n \in N, \qquad \lambda_j = jh, \quad j = 1, 2, \ldots n+1,$$

$$Q_j = Q(\lambda_j), \quad P_j = P(\lambda_j), \qquad u_j = u(\lambda_j), \qquad f_j = f(\lambda_j).$$

• So we can use the following discrete analogue

$$(13)\qquad -\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}-Q_ju_j=f_j,\quad j=1,...,n,$$

$$(14)\qquad v_0=C,\qquad v_{n+1}=D.$$

for the problem (4), (5).

Similarly as in [1] we can define operators W_n^1, W_n^2 from $\mathcal{F}$ to be the solutions of the homogeneous difference equation of equation (13)

$$l(W_n^i)\equiv-\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}-Q_ju_j=0,\qquad i=1,2,\quad j=1,...N-1.$$

$$W_0^1=0,\quad W_1^1=hI,\quad W_N^2=0,\quad W_N^2=hI,$$

and the operator V_n as

$$V_n=\begin{vmatrix} W_n^1 & W_n^2 \\ \frac{W_n^1-W_{n-1}^1}{h} & \frac{W_n^2-W_{n-1}^2}{h}\end{vmatrix}.$$

Taking

$$(15)\qquad G_n^k=\begin{cases}\frac{W_k^2W_n^1}{V(h)} & 0\le n\le k\\ \frac{W_k^1W_n^2}{V(h)} & k\le n\le N\end{cases}$$

similarly as in the classical case it can be proved that (see [1])

$$(16)\qquad l(G_n^k)=\delta_{n-k}h^{-1}I.$$

Since $\delta_{n-k}h^{-1}$ is the discrete analogues for δ function, this means that the equality (16 is the discrete analogue for (7). Therefore the operators G_n^k given by relation (15) can be treated as the discrete Green operator.

From relation (15) it follows that the operators G_n^k can be written in the form

$$(17)\qquad G_n^k=g_n^kI+g_{n,c}^k$$

where g_n^k are the numerical constants and $g_{n,c}^k$ are the operators representing continuous functions, and it holds

$$g_n^k = \begin{cases} \dfrac{w_k^2 w_n^1}{v_0} & 0 \le n \le k \\ \\ \dfrac{w_k^1 w_n^2}{v_0} & k \le n \le N \end{cases}$$

Similarly as in the classical case we can give the solution for the problem (13), (14) as

$$u_n = h \sum_{k=1}^{N-1} G_n^k f_k + \frac{W_n^2}{W_0^2} C + \frac{W_n^1}{W_n^N} D \tag{18}$$

Since the operators f_k, $k = 1, 2, ..., N-1$, and the operators C and D are from $\mathcal{F}_c$ it follows that the operators u_n are from $\mathcal{F}_c$. Further on, we shall show that the operators u_n, given by relation (18) can be treated as the approximate solution for the problem (4), (5) at each point $\lambda = \lambda_i$, $i = 1, 2, ..., N-1$. Namely, instead of operator function $u(\lambda)$ given by (9), which represent continuous function of two variables λ and t, we consider the operators u_n given by (18), which also represent continuous function of variable t.

4. The error of approximation

Let us denote by $\mathcal{F}_{cr}$ a subset of $\mathcal{F}_c$ consisting of continuous real valued functions. If $a = \{a(t)\}$ and $b = \{b(t)\}$ are from $\mathcal{F}_{cr}$, we can say that the operator a is greater than or equal to the operator b, denoted by $a \underline{\geq} b$, if $a(t) \ge b(t)$ for each $t \ge 0$ (see [2], p. 237). If the last inequality holds on the interval $[T_1, T_2]$, then we write $a \underline{\geq}_{[T_1,T_2]} b$.

Analogously, we shall say that $a(\lambda) \underline{\geq}_{[T_1,T_2]} b(\lambda)$, if $a(\lambda)$ and $b(\lambda)$ are operational functions representing continuous real valued functions of two variables, $a(\lambda) = \{a(\lambda, t)\}$, $b(\lambda) = \{b(\lambda, t)\}$, such that $a(\lambda, t) \ge b(\lambda, t)$, for $t \in [T_1, T_2], \lambda \in [0, 1]$.

The absolute value of an operator a from $\mathcal{F}_{cr}$, $a = \{a(t)\}$, denoted by $\mid a \mid$, is the operator $\mid a \mid = \{\mid a(t) \mid\}$. Also, we put $\mid a(\lambda) \mid = \{\mid a(\lambda, t) \mid\}$.

If the operators a and b are from $\mathcal{F}_{cr}$, then

$$| a+b | \leq | a | + | b |,$$

$$| ab |= \int_0^t a(\tau)b(t-\tau)d\tau | \leq | a || b |,$$

and

$$| a | \leq_{[T_1,T_2]} A(T_1,T_2)l, \quad A(T_1,T_2) = \max_{t\in[T_1,T_2]}\{| a(t) |\}.$$

If the operators a and b are not from $\mathcal{F}_{cr}$, but we can write them in a form $a = \alpha_1 I + a_1$ and $b = \beta_1 I + b_1$, where, now, a_1 and b_1 are the operators from $\mathcal{F}_{c_r}$, then it holds

$$| l(\alpha_1 I + a_1) | \leq | \alpha_1 | l + | la_1 | \leq | \alpha_1 | l + l | a_1 | .$$

The operators W_n^1, W_n^2 and the operators g_n^k are not from $\mathcal{F}_c$, but the can be written in a form $\alpha_1 I + \alpha_c$, for each $k = 1, 2, ..., N-1$, where α_1 is a numerical constant and α_c is an operator representing continuous function. Therefore they can be estimated them by using factor l, (integral operator, representing constant function 1) which makes them continuous functions for each λ.

We can show that

$$\max_{0\leq x_n\leq X} |l(W_n^1 - W_1(nh))| \leq Mh^2 l,$$

$$\max_{0\leq x_n\leq X} |l(W_n^2 - W_2(nh))| \leq Mh^2 l,$$

and

$$\begin{aligned} &|l(G_n^k - G(nh,kh))| \\ (19) \qquad &\leq |l(g_n^k - g^1(nh,kh))| + |l(g_{n,c}^k - g_c(nh,kh))| \\ &\leq_{[0,T]} R_1 h^2 l, \end{aligned}$$

where M and R_1 is numerical constants estimating g_n^k, $g^1(nh,kh)$, $g_{n,c}^k$, $g_c(nh,kh)$.

So, the error of approximation for the numerical solution in the field of Mikusiński operators obtained by using operator Green function can be estimated as

$$|(u_n - u(nh)| \le h \sum_{k=1}^{N-1} |(G_n^k - G(nh,kh))f_k| + R_2 h^2 l \le_{[0,T]} R f_M l h^2,$$

where

$$R = \max(R_1, R_2), \quad f_M = \max_{[0,T]} f_k.$$

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