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AN ASYMPTOTICALLY OPTIMAL ALGORITHM FOR CONSTRUCTION OF OPTIMAL DIGITAL STAR-SHAPED POLYGONS

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Abstract

This paper presents an algorithm for construction of a digital star-shaped polygon with given center, which can be inscribed in the square grid of given size and which has the maximal possible number of edges.

It is proved that the proposed algorithm is asymptotically optimal.

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1. Introduction

There is an increasing interest in optimization problems on the integer grid in last few years, as well as in efficient constructions (w.r.t. the required computational time) of their optimal solutions. Such problems have been considered in a number of recent papers (see, for example, [2], [4], [5], [6], [9], [10]).

An algorithm for solving the following problem is proposed: Given the integers p, q, r, such that $0 \leq p, q \leq r$, determine a digital star-shaped polygon ([8]) with the center (p,q), which can be inscribed into the $r \times r$–grid and which has as much edges as possible.

The proposed algorithm is shown to be asymptotically optimal. Some results from [3] are used. These results are related to the solution of the optimization problem of finding a convenient expression for the maximal number of edges of a digital star-shaped polygon inscribed into a square grid of a given size. A brief account on this topic is given in Section 3.

The structure of the algorithm includes the following three levels:

- registration of a vertex
- construction of vertices of the required polygon by one pass through a generalized Farey sequence
- construction of the whole required polygon by calling eight constructions of the second level .

2. Definitions and denotations

Let $S(m,n)$, where $m \leq n$, denote the portion of the integer grid (Fig. 2.), consisting of all digital points (a,b), which satisfy

$$0 < b < a \leq n \quad \text{and} \quad b \leq m \ .$$

The same denotation ($S(m,n)$) is used (Fig. 3.) for the portion of the grid, which is obtained from the above one by applying axial symmetries and rotations for 90^o, 180^o and 270^o around the point $(0,0)$.

Further, let a *polygon of the form* $P(m,n)$ be a polygon included into $S(m,n)$, the vertices of which in anticlockwise order are digital points $(0,0) = A_1$, $(1,0) = A_2$, A_3, ... A_{k-1}, $A_k = (1,1)$, such that the open line segment (A_1, A_i) belongs to the interior of the polygon for $3 \leq i \leq k-1$.

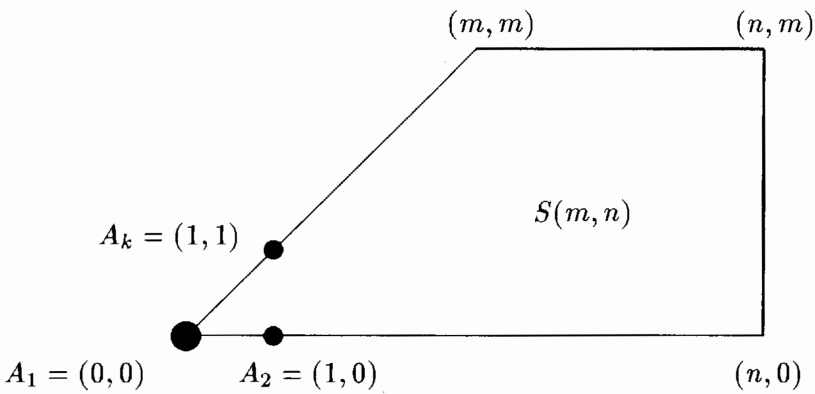

Figure 2. The region $S(m,n)$, for $m \leq n$

A *digital point* is a point with integer coordinates in the coordinate plane.

A *star-shaped polygon* is a polygon P, for which there exists an interior point C with the property that the open line segment CW is included into the interior of P, for each point W of P. Those points of P, which might play the role of the point C – constitute the *nucleus* of P.

A *ray* is a half-line with the beginning point $(0,0)$, which also passes through another point of $S(m,n)$.

A *digital star-shaped polygon* is a star-shaped polygon P, which satisfies the following two conditions:

a) all the vertices of P are digital points

b) there exists a digital point (*center*) in the nucleus of P.

The $m \times n$*–grid* is the set of all digital points (x,y) satisfying $0 \leq x \leq m$ and $0 \leq y \leq n$.

Let natural numbers m and n be given so that $m \leq n$.

The *generalized Farey* (m,n)*-sequence*, denoted by $F(m,n)$, is the strictly increasing sequence of all the fractions of the form b/a, where the integers a and b are relatively prime ($(a,b) = 1$) and satisfy that $b < a \leq n$ and $b \leq m$.

Thus the sequence $F(4,7)$ seems as follows :

$$\frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}.$$

Members of $F(m,n)$ will be denoted by b_i/a_i, for $i = 1, 2, ...$

Let $f(m,n)$ denote the number of members of the sequence $F(m,n)$. An important observation is that the number of rays (= the number of candidates for the vertices of the digital star-shaped polygon) within $S(m,n)$ is equal to $f(m,n)$.

Given the $m \times n$–grid, the point (a_i, b_i) is *prolongible with a factor* k (Fig. 4.) if $k \cdot a_i \leq n$, $k \cdot b_i \leq m$, and at least one of the inequalities $(k+1) \cdot a_i > n$, $(k+1) \cdot b_i > m$ is satisfied. If it is merely said that a point is *prolongible*, then it is meant that the factor k is greater than 1.

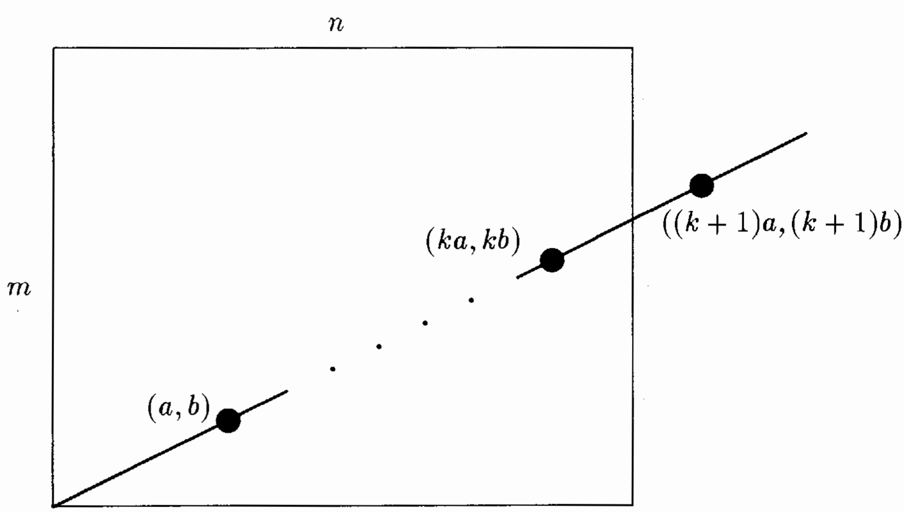

Figure 4. The prolongibility with factor k

It can be shown ([3]) that the construction of an optimal digital star-shaped polygon can be completed by replacing some prolongible points (a_i, b_i) by $(2 \cdot a_i, 2 \cdot b_i)$. It is said in such a case that the second point is obtained by "prolonging".

Collinearity is a triple of collinear points which are *chosen* to be representatives of some three consecutive rays.

3. Some preliminary results

The required construction is equivalent to the construction of eight polygons of the form $P(m,n)$, each one of which has as much vertices as possible. The set of vertices of an optimal digital star-shaped polygon with the center $(0,0)$ can be expressed as the union of the sets of vertices of these eight polygons, without the vertex $(0,0)$. Each one of these polygons should be placed (by applying the translation to the center (p,q), as well as the axial symmetries and/or rotations for $\pi/2$) into an octant (with the central angle equal to $\pi/4$) of the $r \times r-$grid, determined by the center (p,q) (Fig. 3.) . In addition, the resulting optimal digital star-shaped polygon should have as vertices the eight digital points, which are 8-neighbours of the center (p,q).

Each non-central vertex (a_k, b_k) of the optimal polygon of the form $P(m,n)$ corresponds to a member b_k/a_k of the generalized Farey sequence $F(m,n)$ ([1]). The converse is not true, since some three consecutive points (a_{i-1}, b_{i-1}), (a_i, b_i) and (a_{i+1}, b_{i+1}), obtained by this correspondence – may be collinear. The treatment of collinearities has the following three stages:

1. Characterization of collinearities in the sequence (a_i, b_i)
2. Avoiding collinearities by prolonging
3. Discussing the situations in which prolonging leads to new collinearities.

All the assertions below in this section follow from some properties of generalized Farey sequences and can be found in [3].

Let (a_{i-1}, b_{i-1}), (a_i, b_i) and (a_{i+1}, b_{i+1}) be three consecutive vertex candidates corresponded to the required digital star-shaped polygon. If they are collinear, then $q_i = 2$. Such a collinearity cannot be avoided by replacing the point (a_i, b_i) by another point belonging to the same ray. Therefore, one should concentrate on the "neighbours" of the points (a_j, b_j) with $q_j = 2$. Also, at most one of the points (a_{i-1}, b_{i-1}) and (a_{i+1}, b_{i+1}) is prolongible.

This means that a trial should be made to replace the prolongible one (if any) of the points (a_{i-1}, b_{i-1}) and (a_{i+1}, b_{i+1}), say (a_{i+1}, b_{i+1}), with some other point of the form $(r \cdot a_{i+1}, r \cdot b_{i+1})$. The coefficient q_{i+1} must belong to the set $\{1,2,3,4\}$. However, the replacement is not possible for $q_{i+1} = 1$ and $q_{i+1} = 2$. On the other hand, if $q_{i+1} = 3$ and $q_{i+1} = 4$, then r is not greater than 2 (if $q_{i+1} = 3$, then even this replacement is not always possible).

Thus, if the point (a_{i+1}, b_{i+1}) is prolongible, then the only possibility for avoidance the collinearity of the points (a_{i-1}, b_{i-1}), (a_i, b_i) and (a_{i+1}, b_{i+1}) is to replace the point (a_{i+1}, b_{i+1}) by the point $(2 \cdot a_{i+1}, 2 \cdot b_{i+1})$.

It should be emphasized that avoidance the collinearity of some three consecutive points may result in the collinearity of some other three consecutive points.

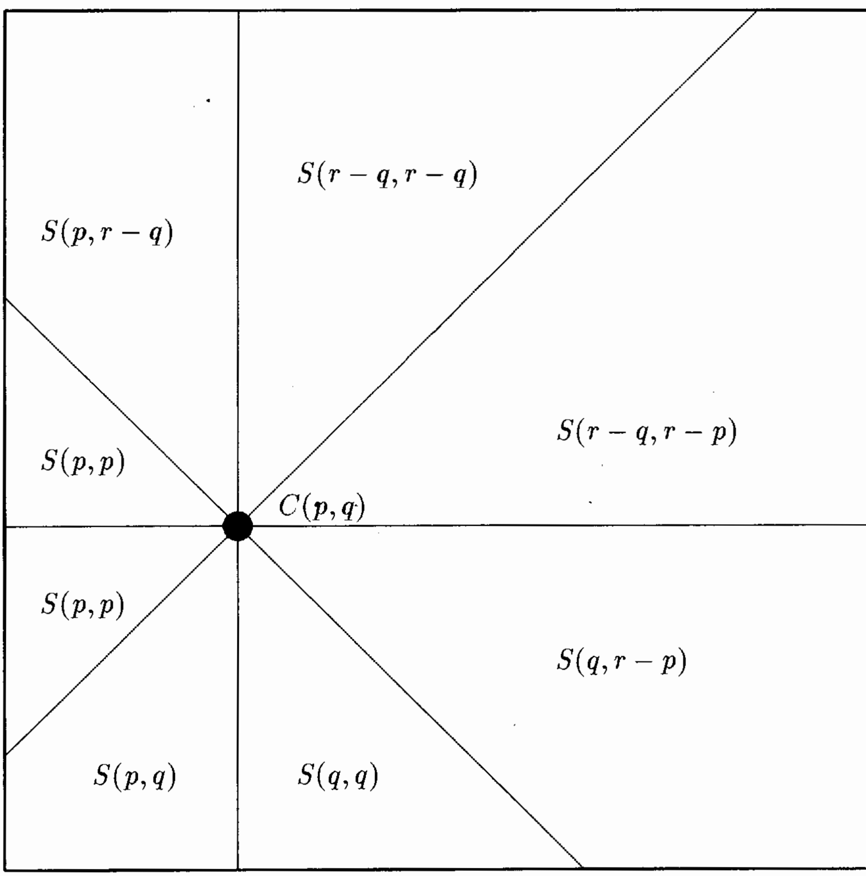

Figure 3. The eight regions of $r \times r$−grid for $p < q$ ard $p < r - q$

In particular, if $q_{i+1} = 4$, then the points (a_i, b_i), $(2 \cdot a_{i+1}, 2 \cdot b_{i+1})$ and (a_{i+2}, b_{i+2}) are necessarily collinear. None of the points (a_i, b_i) and (a_{i+2}, b_{i+2}) is prolongible, so that the new collinearity cannot be avoided in some other way. Also, the collinearities (a_{i-1}, b_{i-1}), (a_i, b_i) (a_{i+1}, b_{i+1}) and (a_{i+1}, b_{i+1}), (a_{i+2}, b_{i+2}), (a_{i+3}, b_{i+3}) are not simultaneously possible; it is consequently not possible in the considered case to avoid two collinearities by making only one new collinearity.

As a conclusion, it follows that the number of collinearities cannot be reduced by replacing the point (a_{i+1}, b_{i+1}) by the point $(2 \cdot a_{i+1}, 2 \cdot b_{i+1})$ when $q_{i+1} = 4$.

If $q_{i+1} = 3$, and the point (a_{i+1}, b_{i+1}) is prolongible, then the same point replacement leads to a reduction of the number of collinearities. Namely, none of the collinearities (a_i, b_i), $(2 \cdot a_{i+1}, 2 \cdot b_{i+1})$, (a_{i+2}, b_{i+2}) and $(2 \cdot a_{i+1}, 2 \cdot b_{i+1})$, (a_{i+2}, b_{i+2}), (a_{i+3}, b_{i+3}) is possible.

If $q_{i+4} = 2$, $q_{i+3} = 3$, and the point (a_{i+3}, b_{i+3}) is analogously replaced by $(2 \cdot a_{i+3}, 2 \cdot b_{i+3})$, then the new collinearity $((2 \cdot a_{i+1}, 2 \cdot b_{i+1}), (a_{i+2}, b_{i+2}), (2 \cdot a_{i+3}, 2 \cdot b_{i+3}))$ would arise iff $q_{i+2} = 1$. Such a situation is not possible.

On the other hand, if $q_i = q_{i+2} = 2$ and $q_{i+1} = 3$, then the replacement of the point (a_{i+1}, b_{i+1}) by the point $(2 \cdot a_{i+1}, 2 \cdot b_{i+1})$ (when possible) reduces the number of collinearities by two.

The algorithm also uses an efficient construction of the successive members of a generalized Farey sequence, which is based on the following statement from [1] :

Let b_1/a_1 and b_2/a_2 denote arbitrary two consecutive members of $F(m,n)$ and let (x_0, y_0) be an integral solution of the equation: $a_1 \cdot x - b_1 \cdot y = 1$. Then b_2/a_2 is determined by the relations: $b_2 = x_0 + rb_1$ and $a_2 = y_0 + ra_1$, where

$$r = min\{\lfloor \frac{n - y_0}{a_1} \rfloor, \qquad \lfloor \frac{m - x_0}{b_1} \rfloor\}.$$

4. Description of the algorithm

Input: natural numbers p, q and r $(0 < p, q < r)$

Output: a digital star-shaped polygon $P(r,p,q)$ with as much vertices as possible, which can be included into the $r \times r$–grid and which has the center (p,q).

The vertices of the polygon $P(r,p,q)$ are constructed by using eight generalized Farey sequences. Each one of these eight sequences is used for the construction of vertices of $P(r,p,q)$ belonging to one of the eight regions of the form $S(m,n)$ and has its own associated *state*, a number from the set $\{1,2,...,8\}$. The arrows showing the directions of producing the vertices of $P(r,p,q)$, as well as the corrersponding states, are depicted in Fig. 5.

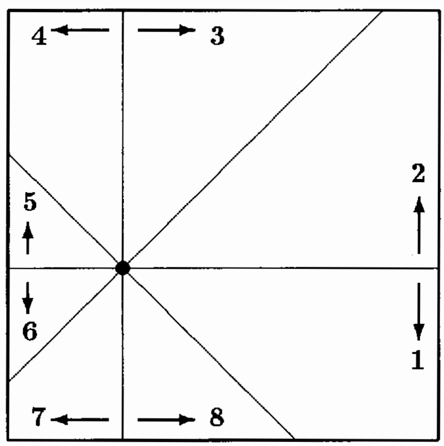

Figure 5. Directions of generation the vertices of $P(r,p,q)$ in states 1,2,...,8

The following function $h : (b, a, PF, p, q, state) \longrightarrow (x,y)$

associates the vertex (x,y) of the polygon $P(r,p,q)$ to a member b/a of one of the eight generalized Farey sequences, depending on the state, prolongibility factor PF and on the position of the center (p,q):

$$x \leftarrow p + PF \cdot x_{dif}(a,b,state); \qquad y \leftarrow q + PF \cdot y_{dif}(a,b,state) ,$$

where the values of $x_{dif}(a,b,state)$ and $y_{dif}(a,b,state)$, depending on the state, are given in Table 1. :

state	1	2	3	4	5	6	7	8
$x_{dif}\ (a, b, state)$	$+a$	$+a$	$+b$	$-b$	$-a$	$-a$	$-b$	$+b$
$y_{dif}\ (a, b, state)$	$-b$	$+b$	$+a$	$+a$	$+b$	$-b$	$-a$	$-a$

Table 1.

A vertex is associated to a member b_2/a_2 of a generalized Farey sequence whenever $q_2 \neq 2$. If $q_2 = 2$, then this is the case only provided the q-value corresponding to one of the two neighbouring members is equal to 3 and that the vertex corresponding to that member is 2-prolongible.

On the other hand, if a member b_i/a_i is "accepted" for the registration of a vertex, then the corresponding vertex coordinates a_i and b_i are in most cases used as coordinates of the vertex. Exceptionally, if the q-value of the considered member is equal to 3, with the 2-prolongible associated point (a_i, b_i) and at least one of the two neighbouring members has the q-value equal to 2, then the coordinates $2 \cdot a_i$ and $2 \cdot b_i$ are used instead.

Generation in order all the vertices of the polygon $P(r,p,q)$ within one of the regions of the form $S(m,n)$ is performed by one call of the following procedure:

PROCEDURE gf(m, n, p, q, state: integer);

BEGIN (* gf *)

(* Initialize the first three members of the sequence $F(m,n)$, as well as the members q_2 and q_3. This initialization is valid whenever $1/(n-2) < 2/n$, that is, whenever $n > 4$, except for q_3, which is proper for $n > 6$: *)

$b_1/a_1 := 1/n;\quad b_2/a_2 := 1/(n-1);\quad b_3/a_3 := 1/(n-2);\quad q_2 := 2;\quad q_3 := 2;$

REPEAT

IF one of the following three conditions is valid:

- $q_2 \neq 2$

- $(q_2 = 2)$ AND $(q_1 = 3)$ AND $(2b_1 \le m)$ AND $(2a_1 \le n)$
- $(q_2 = 2)$ AND $(q_3 = 3)$ AND $(2b_3 \le m)$ AND $(2a_3 \le n)$

THEN IF

$(q_2 = 3)$ AND $(q_1 = 2$ OR $q_3 = 2)$ AND $(2b_2 \le m)$ AND $(2a_2 \le n)$

THEN	Output the vertex $(x, y) = h(b_2, a_2, 2, p, q, state)$
ELSE	Output the vertex $(x, y) = h(b_2, a_2, 1, p, q, state)$;

Generate the following member b_4/a_4 of $F(m,n)$, in accordance with the statement at the end of Section 3.

Move to the next member of $F(m,n)$ by making the following necessary adjustments:

$$b_1/a_1 \longleftarrow b_2/a_2; \quad b_2/a_2 \longleftarrow b_3/a_3; \quad b_3/a_3 \longleftarrow b_4/a_4;$$

$$q_1 \longleftarrow q_2; \quad q_2 \longleftarrow q_3; \quad q_3 \longleftarrow b_4 \cdot a_2 - b_2 \cdot a_4;$$

UNTIL $b_4/a_4 = m/(m+1)$ (* the last member of $F(m,n)$ *)

END; (* gf *)

The main "shell" for the construction of all the vertices of the polygon $P(r,p,q)$ has the following outlook:

BEGIN (* star-shaped *)

Write down the eight vertices of the form

$$(p+i, q+j), \text{ where } (i,j) \in \{-1, 0, 1\} \text{ and } (i,j) \neq (0,0);$$

```
IF  p < q  THEN BEGIN  gf(n-q,n-p, 2);  gf(n-q,n-q, 3);
                       gf( p , p , 6);  gf( p , q , 7)   END
           ELSE BEGIN  gf(n-p,n-p, 2);  gf(n-p,n-q, 3);
                       gf( q,  p,  6);  gf( q,  q,  7)   END;

IF n-q < p THEN BEGIN  gf(n-q,n-q, 4);  gf(n-q, p,  5);
```

```
                   gf(n-p, q,  8);  gf(n-p,n-p, 1)   END
        ELSE BEGIN gf( p, n-q, 4);  gf( p,  p,  5 );
                   gf( q,  q,  8);  gf( q, n-p, 1 )  END;
```

END. (* star-shaped *)

5. Analysis of the algorithm

Let $s(m,n)$ denote the maximal number of vertices of a polygon of the form $P(m,n)$ (which can be included into the region $S(m,n)$, Fig. 2.)

We proceed with two lemmae, which are used for proving the asymptotical optimality of the proposed algorithm.

Lemma 1. $f(m,n) \leq s(4 \cdot m, 4 \cdot n)$.

Proof. There are $f(m,n)$ rays through (0,0) within $S(m,n)$. Each one of these rays contains the point (a_i, b_i), where b_i/a_i is a member of $F(m,n)$. The statement of the lemma is equivalent to the claim that each one of the mentioned rays includes a vertex of a polygon Q of the form $P(4m, 4n)$.

The polygon Q can be constructed in the following manner:

a) **If** the point (a_i, b_i) is not collinear with the points (a_{i-1}, b_{i-1}) and (a_{i+1}, b_{i+1}) , **then** this point is a vertex of Q.

b) Let (a_j, b_j) , (a_{j+1}, b_{j+1}) ,... (a_{j+k}, b_{j+k}) (where $k \geq 2$) be any *maximal* sequence of successive collinear points of the form (a_i, b_i). Vertices V_0 , V_1 , ... , V_k of the polygon Q are associated to that sequence as follows:

If k is even and $k \geq 4$, **then begin**

$$V_i = (a_{j+i},\ b_{j+i}) , \quad \text{for} \quad i = 0, 1, 3, 5, ..., k-1, k$$
$$V_i = (2 \cdot a_{j+i},\ 2 \cdot b_{j+i}) , \quad \text{for} \quad i = 2, 4, 6, \ ...\ , k-2$$

end;

If k is odd and $k \geq 5$, **then begin**

$$V_i = (a_{j+i},\ b_{j+i}), \quad \text{for } i = 0,1,3,5,...,k-4,k-1,k$$
$$V_i = (2 \cdot a_{j+i},\ 2 \cdot b_{j+i}), \quad \text{for } i = 2,4,6, \ldots, k-3, k-2$$

end;

If $k = 3$ **then begin**
  $V_i = (a_{j+i},\ b_{j+i})$ for $i = 0,2,3$;
  If the points $(a_{j-1},\ b_{j-1})$, $(a_j,\ b_j)$ and $(2 \cdot a_{j+1},\ 2 \cdot b_{j+1})$
    are not collinear **then**
      $V_1 = (2 \cdot a_{j+1},\ 2 \cdot b_{j+1})$
  else $V_1 := (3 \cdot a_{j+1},\ 3 \cdot b_{j+1})$
end;
If $k = 2$ **then begin**
  $V_i = (a_{j+i},\ b_{j+i})$ for $i = 0,2$;
  V_1 is the first one among the points
    $(2 \cdot a_{j+1},\ 2 \cdot b_{j+1})$, $(3 \cdot a_{j+1},\ 3 \cdot b_{j+1})$ and $(4 \cdot a_{j+1},\ 4 \cdot b_{j+1})$,
  which does not belong to any one of the lines
  determined by the points $(a_{j-1},\ b_{j-1})$ and $(a_j,\ b_j)$,
  respectively by the points $(a_{j+2},\ b_{j+2})$ and $(a_{j+3},\ b_{j+3})$,
end;

It is obvious that there are no three collinear successive vertices among $V_0,\ V_1, \ldots,\ V_k$, as well as that all vertices of the polygon Q obtained in this manner belong to $S(4m, 4n)$. Since the points (a_i, b_i) are used for $k \geq 4$ with the vertices V_0, V_1, V_{k-1} and V_k, it follows that no new collinearities may arise in the course of avoiding the old ones. The latter condition is also satisfied with the constructions proposed for $k \in \{2,3\}$. □

Let $\Theta(f)$ denote the set of functions g such that there exist positive constants c_1 and c_2 which satisfy $c_1 \cdot f(x) \leq g(x) \leq c_2 \cdot f(x)$ for sufficiently large x. [8]

Lemma 2. $s(m,n) \in \Theta(m \cdot n)$.

It is known ([1]) that the function $f(m,n)$ belongs to $\Theta(m \cdot n)$. Lemma 1. and the obvious inequality

$$s(4 \cdot m, 4 \cdot n) \le f(4 \cdot m, 4 \cdot n)$$

imply that $s(4 \cdot m, 4 \cdot n)$ also belongs to $\Theta(m \cdot n) = \Theta(4m \cdot 4n)$.

This inclusion is equivalent to the statement of the lemma. □

Theorem 1. *The proposed algorithm for the construction of optimal digital star-shaped polygons is asymptotically optimal.*

Proof. Both functions $f(m,n)$ and $s(m,n)$ belong to $\Theta(m \cdot n)$ by [1] and by Lemma 2 respectively. In addition, each member of a generalized Farey sequence can be generated in constant time ([1]) from the previous one. This implies that the number of steps of the proposed algorithm has the same order of magnitude as the number of generated vertices. □

6. An Example

We illustrate the work of the proposed algorithm for $r = 10, p = 3, q = 4$. The constructed optimal digital star-shaped polygon with these parameters is shown in Fig. 6. The avoided collinearities are emphasized in that figure.

The complete trace of the construction is listed below. Each member b_i/a_i of a generalized Farey sequence is represented by the quintuple

$$(\ b_i,\ \ a_i,\ \ q_i,\ \ p + x_{dif}(a_i, b_i, state),\ \ q + y_{dif}(a_i, b_i, state)\ )$$

The coefficients q_i are omitted with the first and the last member of each generalized Farey sequence, since they cannot be determined in these cases. If $q_i = 3$, then the letters "`P`" and "`N`" are used with the prolongible and non-prolongible cases respectively.

The letters "`D`" and "`A`" at the end of a quintuple respectively denote that the "doubled" prolongibility factor is used and that the corresponding vertex is "additional" (obtained by avoidance of a collinearity).

The parameters m and n (the sizes of the corresponding rectangular grid) are listed with each one of the eight states. This is also the case with the state-dependent schemes of the form `(3 + c, 4 + d)` (where $\{\texttt{c}, \texttt{d}\} \subset \{\ +a,\ -a,\ +b,\ -b\ \}$), which are used for calculating the position of the corresponding points (x, y).

```
state 2:   m =   6  n =   7          (3 + a, 4 + b)
( 1, 7, - ,10, 5 )   ( 1, 6, 2 , 9, 5 )   ( 1, 5, 2 , 8, 5 )
( 1, 4, 3N, 7, 5 )   ( 2, 7, 1 ,10, 6 )   ( 1, 3, 4 , 6, 5 )
( 2, 5, 2 , 8, 6 )   ( 3, 7, 1 ,10, 7 )   ( 1, 2, 7 , 5, 5 )
( 4, 7, 1 ,10, 8 )   ( 3, 5, 2 , 8, 7 )   ( 2, 3, 4 , 6, 6 )
( 5, 7, 1 ,10, 9 )   ( 3, 4, 3N, 7, 7 )   ( 4, 5, 2 , 8, 8 )
( 5, 6, 2 , 9, 9 )   ( 6, 7, - ,10,10 )

state 3:   m =   6  n =   6          (3 + b, 4 + a)
( 1, 6, - , 4,10 )   ( 1, 5, 2 , 4, 9 )   ( 1, 4, 2 , 4, 8 )A
( 1, 3, 3P, 4, 7 )D  ( 2, 5, 1 , 5, 9 )   ( 1, 2, 5 , 4, 6 )
( 3, 5, 1 , 6, 9 )   ( 2, 3, 3P, 5, 7 )D  ( 3, 4, 2 , 6, 8 )A
( 4, 5, 2 , 7, 9 )   ( 5, 6, - , 8,10 )

state 6:   m =   3  n =   3          (3 - a, 4 - b)
( 1, 3, - , 0, 3)    ( 1, 2, 3N, 1, 3 )   ( 2, 3, - , 0, 2 )

state 7:  m =   3  n =   4           (3 - b, 4 - a)
( 1, 4, - , 2, 0 )   ( 1, 3, 2 , 2, 1 )A  ( 1, 2, 3P, 2, 2 )D
( 2, 3, 2 , 1, 1 )A  ( 3, 4, - , 0, 0 )

state 4:  m =   3  n =   6           (3 - b, 4 + a)
( 1, 6, - , 2,10 )   ( 1, 5, 2 , 2, 9 )   ( 1, 4, 2 , 2, 8 )A
( 1, 3, 3P, 2, 7 )D  ( 2, 5, 1 , 1, 9 )   ( 1, 2, 5 , 2, 6 )
( 3, 5, 1 , 0, 9 )   ( 2, 3, 3N, 1, 7 )   ( 3, 4, - , 0, 8 )

state 5:   m =   3  n =   3          (3 - a, 4 + b)
( 1, 3, - , 0, 5 )   ( 1, 2, 3N, 1, 5 )   ( 2, 3, 0 , 0, 6 )

state 8:   m =   4  n =   4          (3 + b, 4 - a)
( 1, 4, - , 4, 0 )   ( 1, 3, 2 , 4, 1 )A  ( 1, 2, 3P, 4, 2 )D
( 2, 3, 2 , 5, 1 )A  ( 3, 4, - , 6, 0 )

state 1:  m =   4  n =   7           (3 + a, 4 - b)
( 1, 7, 0 ,10, 3 )   ( 1, 6, 2 , 9, 3 )   ( 1, 5, 2 , 8, 3 )
( 1, 4, 3N, 7, 3 )   ( 2, 7, 1 ,10, 2 )   ( 1, 3, 4 , 6, 3 )
( 2, 5, 2 , 8, 2 )   ( 3, 7, 1 ,10, 1 )   ( 1, 2, 7 , 5, 3 )
( 4, 7, 1 , 10,0 )   ( 3, 5, 2 , 8, 1 )A  ( 2, 3, 3P, 6, 2)D
( 3, 4, 2 , 7, 1 )A  ( 4, 5, - , 8, 0 )
```

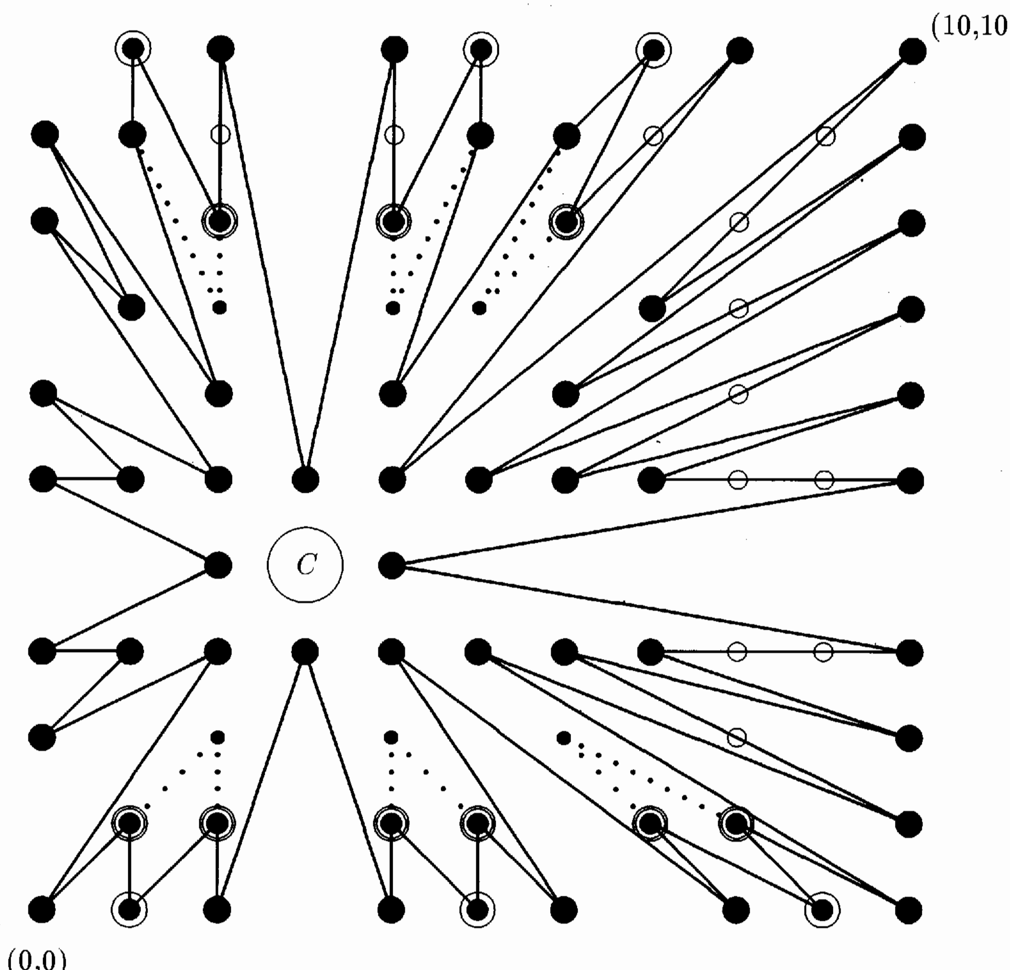

Figure 6. The polygon $P(10,3,4)$

new vertices obtained by avoidance of a collinearity

a vertex obtained by prolonging

. . . edges which would exist if the collinearities were not avoided

a vertex candidate which cannot be accepted due to a collinearity

a vertex which exists disregarding collinearities

C center

an avoided vertex due to prolonging

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