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ON AN EMBEDDING OF THE CONGRUENCE LATTICE INTO THE SUBALGEBRA LATTICE

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Abstract

Weak congruence lattice of an algebra $\mathcal{A}$ is used to establish an embedding of the congruence lattice into the subalgebra lattice, since both $Con\mathcal{A}$ and $Sub\mathcal{A}$ can be taken as sublattices of the former. Algebras having this property are investigated.

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1. Preliminaries

A **weak congruence lattice** $Cw\mathcal{A}$ of an algebra $\mathcal{A}$ is an algebraic lattice consisting of all congruences on all the subalgebras of $\mathcal{A}$.

The diagonal relation $\Delta = \{(x,x) \mid x \in A\}$ is a codistributive element of that lattice: for all $\rho, \theta \in Cw\mathcal{A}$,

$$(\rho \vee \theta) \wedge \Delta = (\rho \wedge \Delta) \vee (\theta \wedge \Delta).$$

$Con\mathcal{A}$ is the filter $[\Delta)$ in $Cw\mathcal{A}$, and for every subalgebra $\mathcal{B}$ of $\mathcal{A}$, $Con\mathcal{B}$ is the interval $[\Delta_{B^2}, B^2]$ ($\Delta_{B^2} = \{(x,x) \mid x \in B\}$), and also an equivalence

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class of the congruence on $C w\mathcal{A}$ induced by the mapping $\rho \rightarrow \rho \wedge \Delta$ ($\rho \in C w\mathcal{A}$).

$Sub\mathcal{A}$ is isomorphic with the ideal $(\Delta]$ under $\mathcal{B} \rightarrow \Delta_{B^2}$.

$\mathcal{A}$ is said to have the **congruence intersection property (CIP)**, if Δ is a distributive element in the lattice $C w\mathcal{A}$, i.e. if for all $\rho, \theta \in C w\mathcal{A}$

$$(\rho \wedge \theta) \vee \Delta = (\rho \vee \Delta) \wedge (\theta \vee \Delta).$$

$\mathcal{A}$ has the **infinite CIP (*CIP)** if Δ is an infinitely distributive element of the lattice $C w\mathcal{A}$, i.e. if for every collection $\{\rho_i \mid i \in I\}$ of weak congruences

$$\Delta \vee \bigwedge_{i \in I} \rho_i = \bigwedge_{i \in I} (\Delta \vee \rho_i).$$

It was proved in [2] that $\mathcal{A}$ has the *CIP if and only if the set $\{\rho \mid \rho \vee \Delta \geq \theta\}$ has a least element for every $\theta \in Con\mathcal{A}$.

Weak congruence lattice was induced in [4], and investigated in [2], [3] and [5] (see also references given there). Connections among congruences and subalgebras were considered by many authors. Here we use the results from [1] and [2].

2. Results

Taking $Con\mathcal{A}$ and $Sub\mathcal{A}$ (the latter up to the isomorphism) as the sublattices of $C w\mathcal{A}$, we shall be concerned with the following problem:

Find conditions under which the mapping $f : Con\mathcal{A} \rightarrow Sub\mathcal{A}$, such that for $\theta \in Con\mathcal{A}$

$$f(\theta) = \underline{\theta} \wedge \Delta$$

is an embedding, where

$$\underline{\theta} = \bigwedge \{\rho \in C w\mathcal{A} \mid \rho \vee \Delta \geq \theta\},$$

all in the lattice $C w\mathcal{A}$. Moreover, every subalgebra of $\mathcal{A}$ should have the same property.

Obviously, f has to be well defined, it has to be an injection and a homomorphism. We shall examine each of these conditions.

Proposition 1. *f is well defined if and only if the algebra $\mathcal{A}$, as well as each of its subalgebras has the *CIP.*

Proof. Obvious, since by [2] (see Preliminaries), for every $\theta \in Con\mathcal{A}$, $\underline{\theta}$ exists if and only if $\mathcal{A}$ has the *CIP. □

Note that in the presence of the *CIP, for every $\theta \in Con\mathcal{A}$,

$$\underline{\theta} = \{\rho \in Con\mathcal{A} \mid \rho \vee \Delta = \theta\}.$$

Proposition 2. *If f is well defined, then $\mathcal{A}$ has an one-element minimal subalgebra $\mathcal{A}_m$.*

Proof. Since $f(\Delta) = \underline{\Delta} \wedge \Delta$, $\underline{\Delta}$ has to be a subalgebra of $\mathcal{A}$. Obviously, $|\underline{\Delta}| = 1$. □

Remark. $\underline{\Delta}$ could be the empty set as well, but only as the minimal element of the lattice $Sub\mathcal{A}$, and not as a subalgebra of $\mathcal{A}$.

Corollary 1. *If f is an injection, then every congruence of $\mathcal{A}$ has exactly one class which is a subalgebra.*

Proof. Obviously, for $\theta \in Con\mathcal{A}$, the class containing the least subalgebra (which by Proposition 2 has one element) is also a subalgebra of $\mathcal{A}$. □

Proposition 3. *If f is an injection, then $\mathcal{A}$ has $\mathcal{A}_m$-regular congruences.*

Proof. Suppose that ρ and θ are two different congruences on $\mathcal{A}$ with the same class - subalgebra $\mathcal{B}$ of $\mathcal{A}$. We can suppose that $\rho \subset \theta$ (otherwise, take $\rho \cap \theta$ and ρ). Consider the congruence $B^2 \vee \Delta$. Obviously, $B^2 \vee \Delta < \theta$. Let $\underline{\theta} \in Con\mathcal{C}$, $\mathcal{C} \in Sub\mathcal{A}$. Thus, $\underline{\theta} \wedge \Delta = C$. Now, $\underline{C^2 \vee \Delta} \leq C^2$, since $\underline{C^2 \vee \Delta}$ is the least element of the class to which C^2 belongs. Hence,

$$(\underline{C^2 \vee \Delta}) \wedge \Delta \leq C^2 \wedge \Delta = C.$$

Now, we have that $f(C^2 \vee \Delta) = f(\theta) = C$, and hence $C^2 \vee \Delta = \theta$, since f is an injection. But B is a class in θ, contradicting the fact that $C^2 \subset \theta$, and $B \subset C$. □

Lemma 1. *If $\mathcal{A}$ has the *CIP, then for $\rho, \theta \in Con\mathcal{A}$,*

$$\underline{\rho \vee \theta} = \underline{\rho} \vee \underline{\theta}.$$

Proof. By the *CIP the least elements exist. Since $(\underline{\rho \vee \theta}) \vee \Delta = \rho \vee \theta$, and $\underline{\rho} \vee \underline{\theta} \vee \Delta = (\underline{\rho} \vee \Delta) \vee (\underline{\theta} \vee \Delta) = \rho \vee \theta$, it follows that $\underline{\rho \vee \theta} \leq \underline{\rho} \vee \underline{\theta}$.

On the other hand, the least elements of the congruence classes obey the lattice order:

$$\underline{\rho} \leq \underline{\rho \vee \theta}, \;\; \underline{\theta} \leq \underline{\rho \vee \theta} \;\text{ whence }\; \underline{\rho} \vee \underline{\theta} \leq \underline{\rho \vee \theta}. \;\square$$

Lemma 2. *If $\mathcal{A}$ has $\mathcal{A}_m$-regular congruences, then every equivalence class on $Cw\mathcal{A}$ induced by the mapping $\rho \mapsto \rho \vee \Delta$ ($\rho \in Cw\mathcal{A}$) contains at least one square.*

Proof. For $\theta \in Con\mathcal{A}$, $[\mathcal{A}_m]_\theta \in Sub\mathcal{A}$, and thus subalgebra uniquely determines a congruence $[\mathcal{A}_m]_\theta \vee \Delta$ on $\mathcal{A}$, which is obviously θ. $\square$

Lemma 3. *If $\mathcal{A}$ as well as all subalgebras of $\mathcal{A}$ have $\mathcal{A}_m$-regular congruences, then minimal elements in an equivalence class induced by $\rho \mapsto \rho \vee \Delta$ ($\rho \in Cw\mathcal{A}$) (if they exist) are squares.*

Proof. Let ρ_m be a minimal element in the class $\{\rho \in Cw\mathcal{A} \mid \rho \vee \Delta = \theta\}$, for $\theta \in Con\mathcal{A}$ and let $\rho_m \in Con\mathcal{B}$ ($\mathcal{B} \in Sub\mathcal{A}$). By $\mathcal{A}_m$-regularity applied on $\mathcal{B}$ and by Lemma 2, there is a square in $\{\beta \in Cw\mathcal{A} \mid \beta \vee d_{B^2} = \rho_m\}$, say C^2 ($C \in Sub\mathcal{A}$). But since $C^2 \vee \Delta = (C^2 \vee d_{B^2}) \vee \Delta = \rho_m \vee \Delta$, it follows that $C^2 = \rho_m$. $\square$

Corollary 2. *If an algebra $\mathcal{A}$ with $\mathcal{A}_m$-regular congruences on all subalgebras has the *CIP, then for $\theta \in Con\mathcal{A}$, $\underline{\theta}$ is a square.*

Proof. $\underline{\theta}$ is a minimal element in the class

$$\{\rho \in Cw\mathcal{A} \mid \rho \vee \Delta = \theta\}. \;\square$$

Proposition 4. *Let $\mathcal{A}$ and all subalgebras of $\mathcal{A}$ have $\mathcal{A}_m$-regular congruences and satisfy the *CIP. Now, for all $\rho, \theta \in Con\mathcal{A}$*

$$\text{if } \; (\underline{\rho \wedge \theta}) \wedge \Delta = (\underline{\rho} \wedge \underline{\theta}) \wedge \Delta \; \text{ then } \; \underline{\rho \wedge \theta} = \underline{\rho} \wedge \underline{\theta}.$$

Proof. Obviously, by the *CIP, $\underline{\rho \wedge \theta} \leq \underline{\rho} \wedge \underline{\theta}$. By Corollary 2, $\underline{\rho \wedge \theta}$ is a square, and since $(\underline{\rho \wedge \theta}) \wedge \Delta = (\underline{\rho} \wedge \underline{\theta}) \wedge \Delta$, $(\underline{\rho} \wedge \underline{\theta}$ is a congruence on the same subalgebra as $\underline{\rho \wedge \theta}$, therefore $\underline{\rho \wedge \theta} = \underline{\rho} \wedge \underline{\theta}$. □

Now, we are ready to give the necessary and sufficient conditions under which f is an embedding.

Theorem 1. *Let $\mathcal{A} \in \mathcal{K}$, where $\mathcal{K}$ is a class of algebras closed under formation of subalgebras. Then, the mapping $f : Con\mathcal{A} \rightarrow Sub\mathcal{A}$ given by $f(\theta) = \underline{\theta} \wedge \Delta$ $(\theta \in Con\mathcal{A})$ is an embedding (in $Cw\mathcal{A}$) if and only if $\mathcal{A}$ satisfies the *CIP, it has $\mathcal{A}_m$-regular congruences on all subalgebras, and the least elements in the classes induced by $\rho \mapsto \rho \vee \Delta$ $(\rho \in Cw\mathcal{A})$ form a sublattice of $Cw\mathcal{A}$.*

Proof. Suppose that f is an embedding. Then $\mathcal{A}$ has the *CIP by Proposition 1. It has $\mathcal{A}_m$-regular congruences on all subalgebras by Proposition 3 (applied on all subalgebras of $\mathcal{A}$). The least elements in the classes induced by $\rho \mapsto \rho \vee \Delta$ form a sublattice of $Cw\mathcal{A}$ by Proposition 4, since by homomorphism $f(\rho \wedge \theta) = f(\rho) \wedge f(\theta)$ i.e. $(\underline{\rho \wedge \theta}) \wedge \Delta = (\underline{\rho} \wedge \underline{\theta}) \wedge \Delta$.

On the other hand, if $\mathcal{A}$ satisfies the above mentioned properties, then f is an embedding. Indeed, f is well defined by the *CIP. f is an injection by Corollary 2: for $\rho, \theta \in Con\mathcal{A}$, $\underline{\rho}$ and $\underline{\theta}$ are squares, obviously different, if $\rho \neq \theta$. Hence, $\underline{\rho} \wedge \Delta \neq \underline{\theta} \wedge \Delta$. By Lemma 1, $f(\rho \vee \theta) = f(\rho) \vee f(\theta)$, and the dual is satisfied by the condition of the theorem. □

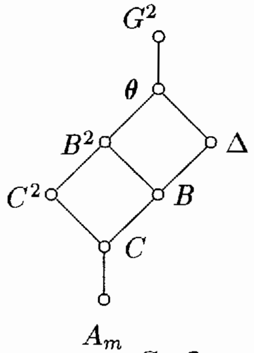
Example 1.

The following groupoid $\mathcal{G}$ satisfies the conditions of the previous theorem and its congruence lattice is embeddable into the subalgebra one, as shown in the diagram of the corresponding weak congruence lattice.

$*$	a	b	c	d
a	a	a	c	d
b	b	a	b	d
c	a	c	b	d
d	a	c	a	b

$\mathcal{G} = (\{a, b, c, d\}, *)$.

$\theta : \{\{a,b,c\},\{d\}\};\ G = \{a,b,c,d\};\ B = \{a,b,c\};\ C = \{a,b\};\ A_m = \{a\}.$



CwG

The set of minimal elements in the classes induced by the mapping $\rho \mapsto \rho \vee \Delta$ ($\rho \in Cw\mathcal{A}$), plays a particular role in the connection of $Con\mathcal{A}$ and $Sub\mathcal{A}$. Under the conditions of the above theorem this is a set of some squares of subalgebras of $\mathcal{A}$. If this set consists of all squares, then we have the following proposition.

Recall that a variety $\mathcal{V}$ is $\mathcal{A}_m$-regular, if every algebra in this variety is $\mathcal{A}_m$-regular.

Theorem 2. *Let $\mathcal{A} \in \mathcal{V}$, where $\mathcal{V}$ is an $\mathcal{A}_m$-regular variety. Now, if $\mathcal{A}$ has the *CIP, and for every $\mathcal{B} \in Sub\mathcal{A}$, B^2 is a minimal element of some class $\rho \mapsto \rho \vee \Delta$.*

Then, $\mathcal{A}$ is Hamiltonian, and $Sub\mathcal{A} \cong Con\mathcal{A}$.

Proof. Since $\mathcal{A}$ is $\mathcal{A}_m$-regular, the set of squares is a sublattice of $Cw\mathcal{A}$ ([2]). By the condition of the theorem all the squares are minimal elements of the mentioned classes and the embedding from Theorem 1 is an isomorphism.

$\mathcal{A}$ is Hamiltonian by Theorem 16 in [2]. □

An obvious example of the above theorem are Hamiltonian groups in the variety of groups.

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