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ON NUMERICAL SOLVING SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS BY SPLINE IN TENSION

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Abstract

The spline in tenslon is applied to a self-adjoint singularly perturbed two point boundary value problem. The tridiagonal difference scheme generated has the nodal error bounded by Mh^4 for fixed ϵ and by $M\sqrt{\epsilon}$ when $\epsilon \leq h^2$ where M is a constant independent of the mesh size h and perturbation parameter ϵ.

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1. Introduction

We consider the following two-point boundary value problem:

$$(1)\begin{cases} Ly = -\epsilon y" + p(x)y = f(x), x \in (0,1) \\ y(0) = \alpha_0, y(1) = \alpha_1. \end{cases}$$

where ϵ is a parameter in $(0,1], \alpha_0, \alpha_1$ are given constants, the functions p and f are in $C^2[0,1]$ and $p(x) \geq p > 0$.

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Finite difference scheme for the problem (1) which are uniformly in ϵ accurate have been examined by many authors ([2)]. Hegarty et al. [3] and Nijima [5] produced uniformly second order difference schemes. Boglaev [1] examined problem in finite element framework and achieved first order accuracy at the nodes. O Riordan and Stynes [6] used finite element method with exponential basis elements and obtained uniform second order acccurcy at the nodes and uniform first-order accuracy between the nodes. Surla [9] and Uzelac, Surla [17] derived uniformly convergent schemes using cubic and quadratic splines and achieved uniform second order accuracy under the conditions $p'(0) = p'(1) = 0$ and first order otherwise. Difference schemes dirived in mentioned paper have a tridiagonal form and they achieved a second order accuracy for fixed ϵ (classical convergence). The scheme derived by Surla and Uzelac [16] (using a suitable combination of the schemes from [2] and from [9]) has a nodal error bounded by $Mh^4/(\epsilon + h^2)$ for $p'(0) = p'(1) = 0$ and by $Mh^4/(\epsilon^{3/2} + h^3)$ otherwise. Vulanović [18], Herceg [4] and Schiskin [8] examined non uniform mesh which must be appropriately chosen and which depends on ϵ. The convergence results are obtained for various difference schemes. The maximal order of convergence iz obtained by Herceg, i.e. a nodal error is bounded by Mn^{-4}, where n is a number of the mesh points. Surla [11], Surla, Herceg, Cvetković [13] and Surla, Stojanović [14] dirived some schemes having the optimal order accuracy in sense of [2], i.e. the schemes have the nondal error bounded by M min $(h, \sqrt{\epsilon})$, $Mmin(h^2, \epsilon)$, or $Mhmin(h, \sqrt{\epsilon})$. In this paper we derive a sheme on uniform mesh which has a forth order accuracy for fixed ϵ and which convergens in respect to ϵ. The scheme is a suitable combination of the two schemes derived in [13].

2. Construction of the scheme

Let be given the uniform mesh $x_i = ih, h = 0(1)n+1, h = 1/(n+1)$. The spline in tension from $C^2[0,1]$ has the from

$$e(x) = e_j = v_{j+1}t + v_j(1-t) + \tag{2}$$

$$+\frac{d_{j+1}}{\rho_j^2}\Big(\frac{sh\mu_j t}{sh\mu_j} - t\Big) + \frac{d_j}{\rho_j^2}\Big(\frac{sh\mu_j(1-t)}{sh\mu_j} - (1-t)\Big)$$

where $\mu_j = \rho_j h, \;\; t = \dfrac{x - x_j}{h}, \; x \in [x_j, x_{j+1}]$.

It is known that $e_j(x) \in$ span $\{1, x, \exp(-\rho_j x), exp(\rho_j x)\}$. The values ρ_j are tension parameters which will be determined. In [13] the unknown coefficients $d_j = e''(x_j)$ and $d_{j+1} = e''(x_{j+1})$ are determined so that the function $e(x)$ satisfies the "comparsion" problem:

$$
(3) \qquad \begin{cases} -\epsilon e''(x) + \tilde{p}(x) e(x) = \tilde{f}(x), \\ e(0) = \alpha_0, \quad e(1) = \alpha_1, \end{cases}
$$

at the grid points. Hare $\tilde{p(x)}$ end $\tilde{f(x)}$ are piecewise polinomial approximations to $p(x)$ and $f(x)$. The obtained schemes have the from

$$
(4) \qquad Rv_j = Qf_j, \quad j = 1(1)n,
$$

where

$$
Rv_j = r^- v_{j-1} + r^c v_j + r^+ v_{j-1},
$$

$$
Qf_j = q^- f_{j-1} + q^c f_j + q^+ f_{j+1}, \quad v_0 = \alpha_0, \; v_{n+1} = \alpha_1.
$$

Let index j be fixed. When $\tilde{p}(x) = p(x_j) = p_j$ and $\tilde{f}(x) = f(x_j) = f_j$ for $x \in [x_{j-1}, x_{j+1}]$ the ecplicit scheme iz derived. The spline belongs to $C^1[0,1]$ and scheme has the form (4) where

$$
(5) \qquad \begin{cases} r^- = r^+ = -\dfrac{\mu_j}{sh\mu_j}, \; r^c = 2\mu_j cth\mu_j, \rho_j = \sqrt{p_j/\epsilon}, \mu_j = \rho_j h, \\ \\ q^c = \dfrac{2\mu_j}{p_j}(cth\mu_j - \dfrac{1}{sh\mu_j}), \quad q^- = q^+ = 0. \end{cases}
$$

When $\tilde{p}(x) = \dfrac{p_j - p_{j-1}}{h}(x - x_{j-1}) + p_{j-1}$ for $x \in [x_{j-1}, x]$ the implicit scheme is obtained. The scheme is derived in [14] in some other way. The coeficients in the scheme (4) have the form

$$
(6) \qquad \begin{cases} r^- = -\dfrac{\mu_{j-1}}{sh\mu_{j-1}}, r^+ = -\dfrac{\mu_{j+1}}{sh\mu_{j+1}}, r^c = 2\mu_j cth\mu_j, \\ \\ q^- = \dfrac{1}{p_{j-1}}\Big(1 - \dfrac{\mu_{j-1}}{sh\mu_{j-1}}\Big), \quad q^+ = \dfrac{1}{p_{j+1}}\Big(1 - \dfrac{\mu_{j+1}}{sh\mu_{j+1}}\Big), \\ \\ q^c = \dfrac{2}{p_j}(1 - \mu_j cth\mu_j). \end{cases}
$$

The corresponding spline belongs to $C^2[0,1]$. Since for $\rho_j \to 0, j=(1)n$, the subic spline arise, by folloving idea from [16] we add scheme (5) to scheme (6).The obtained scheme has the form (4) where

$$r^- = -s(\mu_{j-1}) - s(\mu_j),\ \ r^+ = -s(\mu_{j+1}) - s(\mu_j),\ \ r^c = 4c(\mu_j),$$

(7)

$$q^- = \frac{1}{p_{j-1}}(1-s(\mu_{j-1})),\ q^+ = \frac{1}{p_{j+1}}(1-s(\mu_{j+1})),\ q^c = \frac{2}{p_j}(2c(\mu_j)-s(\mu_j)-1)$$

where

$$s(x) = x/sh(x),\ \ c(x) = xcth(x),\ \ \rho_j = \sqrt{p_j/\epsilon},\ \ \mu_j = \rho_j h.$$

3. Estimating the nodal error

The following lemma bounds the behavoir of the solution $y(x)$ of (1) and it is used in the error analysis.

Lemma 1. ([2], [18]). *Then the solution of (1) can be writen in the form*

$$y(x) = u(x) + w(x) + g(x),$$

where

$$u(x) = q_0 \exp(-x(p(0)/\epsilon)^{1/2}),$$
$$w(x) = q_1 \exp(-(1-x)(p(1)/\epsilon)^{1/2}),$$

q_0, q_1 *are bounded function of* ϵ *independent of* x *and*

(8) $$\mid g^{(i)}(x) \mid \leq M(1+\epsilon^{(1-i)/2}),\ i = 0(1)4.$$

If besides that $p'(0) = p'(1) = 0$, *then*

(9) $$\mid g^{(i)}(x) \mid \leq M(1+\epsilon^{1-(1/2)i}),\ \ i = 0(1)4,$$

M *is a constant independent of* ϵ□.

Throughout the paper M (or δ) denotes different positive constants independent of ϵ and h.

Theorem 1. *Let $y(x) \in C^6[0,1]$ and let $v_j, j = 0(1)n+1$, be the approximation to the solution $y(x)$ at the grid points obtained using (4), (7). Then there is a constsnt M independent of ϵ and h such that for $j = 0(1)n+1$ the estimate*

$$(10) |y(x_j) - v_j| \leq \begin{cases} Mh^3 \min(h, \sqrt{\epsilon}/(\epsilon + h^2) & \text{when } p'(0) = p'(1) = 0, \\ Mh^3 \min(h, \sqrt{\epsilon})/(\epsilon^{3/2} + h^3) & \text{otherwise.} \end{cases}$$

holds.

Proof. The local funcation error $\tau_j(\varphi)$ of the scheme (7) for arbitrary sufficiently smooth function $\varphi(x)$ is defined by

$$\tau_j(\varphi) = R\varphi_j - Q(L_\varphi)_j.$$

The scheme (4) (7) can be writen in the matrix from

$$Av = F$$

where A is a matrix of the system (4) v and F are corresponding vectors. Since, $R(y(x_j) - v_j) = \tau_j(y)$ we have

$$\max_j |y(x_j) - v_j| \leq \|A^{-1}\| \max_j |\tau_j(y)|.$$

Thus, we shall estimate the norm of the matrix and then truncation error. Since $r^- < 0, r^+ < 0$ and $\Delta_j = r^c + r^- + r^+ > 0$ we obtain that

$$(11) \qquad \|A^{-1}\| \leq \max_j |\Delta_j^{-1}| \leq \begin{cases} M\epsilon/h^2 & \text{for } h^2 \leq \epsilon, \\ M\sqrt{3}/h & \text{for } h^2 \geq \epsilon. \end{cases}$$

The Taylor development of $\tau_j(y)$ is given by

$$\tau_j(y) = T_{0j}y_j + T_{1j}y_j^I + T_{2j}y_j^{II} + T_{3j}y_j^{III} + T_{4j}y_j^{IV} + T_{5j}y_j^V + R_j(y)$$

where $T_{oj} = T_{1j} = 0$ and

$$T_{2j} = h^2(r^- + r^+)/2 + \epsilon(q^- + q^c + q^+) - h^2(p_{j-1}q^- + p_{j+1}q^+)/2$$

$$T_{\nu j} = h^\nu(r^+ + (-1)^\nu r^- - \epsilon\nu(\nu-1)(q^+ + (-1)^\nu q^-)/h^2 -$$
$$-(q^+ p_{j+1} + (-1)^\nu q^- p_{j-1}))/\nu!,$$

$\nu = 3, 4, 5$. The expresion $R_j(y)$ contains the remainder terms.

According to Lemma 1 we have

$$\tau_j(y) = \tau_j(u) + \tau_j(w) + \tau_j(g).$$

and we shall separately estimate the truncation errors for the functions u, w and g. Let $h^2 \le \epsilon$. Since $|r^-|, |r^+| \le M$ and $|q^-|, |q^+| \le Mh^2/\epsilon$ from Lemma 1 we obtain that $|R_j(g)| \le Mh^6/\epsilon^{5/2}$. After some Taylor expansions we have that

$$T_{2j} = T_{2j,1} + T_{2j,2}, \quad \text{where}$$

$$T_{2j,1} = -h^2 s(\mu_j) + 2\epsilon(c(\mu)j) - s(\mu_j))/p_j = h^4 p_j/(12\epsilon) + N_1,$$

$$T_{2j,2} = -h^2(s(\mu_{j-1}) + s(\mu_{j+1})) + \epsilon(q^- + q^+ + q^c - c(\mu_j) + s(\mu_j))-$$

$$-h^2(p_{j-1}q^- + p_{j+1}q^+)/2 = -h^4 p_j/(12\epsilon) + N_2, \ |N_i| \le Mh^6/\epsilon^2, \ i = 1, 2;$$

$$|T_{\nu j}| \le Mh^6/\epsilon, \ \ \nu = 3, 4; \ \ |T_{5j}| \le Mh^6.$$

Thus, $|\tau_j(g)| \le Mh^6/\epsilon^2$ when $p'(0) = p'(1) = 0$ and $|\tau_j(g)| \le Mh^6/\epsilon^{5/2}$ otherwise. From (11) we can conclude that the contribution to the error from the function $g(x)$ satisfies (10).

Since $\tau_j(u) = 0$ when $p(x) = p(0) = const.$ we have that $\tau_j(u) = \tilde{\tau}_j(u) - \tau_j(u)$, where $\tilde{\tau}_j(u)$ denotes $\tau_j(u)$ for $p(x) = p(0) = const.$ After Taylor developments about $x = 0$ we obtain

$$|\tau_j(u) - \tilde{\tau}_j(u)| \le Mh^6\epsilon^{-3}x_j^2 u_j \le$$

$$\le Mh^6\epsilon^{-2}\exp(-\delta x_j/\sqrt{\epsilon}) \quad \text{when} \quad p'(0) = p'(1) = 0,$$

$$|\tau_j(u) - \tilde{\tau}_j(u)| \le Mh^6\epsilon^{-3}x_j u_j \le Mh^6\epsilon^{-5/2}\exp(-\delta x_j/\sqrt{\epsilon}) \ \text{ otherwise},$$

δ is a constant independent of ϵ and h. Thus, one finds that contribution to the error from this term satisfies (10).

Similarly,

$$|\tau_j(w) - \tilde{\tau}_j(w)| \le Mh^6\epsilon^{-3}(1 - x_j)^2 w_j \le$$

$$\le Mh^6\epsilon^{-2}\exp(-\delta(1 - x_j)/\sqrt{\epsilon}) \quad \text{for} \quad p'(0) = p'(1) = 0,$$

$$|\tau_j(w) - \tilde{\tau}_j(w)| \le Mh^6\epsilon^{-3}(1 - x_j)w_j \le Mh^6\epsilon^{-5/2}\exp(-\delta(1 - x_j)/\sqrt{\epsilon}) \ \text{ otherwise},$$

where $\tilde{\tau}_j(w)$ denotes $\tau_j(w)$ when $p(x) = p(1) = const.$, $\tilde{\tau}_j(w) = 0$. In that way we can see that Theorem 1 holds for $h \le \epsilon^2$.

Let $h \ge \epsilon^2$. In the case $p'(0) = p'(1) = 0$ we use the form

$$(12) \quad \tau_j(g) = h^2(r^- g''(\sigma_1) + r^+ g''(\sigma_2) - p_{j-1}q^- g''(\sigma_3) - p_{j+1}q^+ g''(\sigma_4))/2$$

$+\epsilon(q^- g''(x_{j-1}) + q^+ g''(x_{j+1}) + q^c g''(x_j)), \;\; x_{j-1} < \sigma_1, \sigma_3 < x_j < \sigma_2, \sigma_4 < x_{j+1}.$

Since

$$|r^-|, |r^+|, |q^-|, |q^+| \leq M, \;\; |q^c| \leq Mh/\sqrt{\epsilon}.$$

from Lemma 1 we have

$$|\tau_j(g)| \leq Mh^2.$$

In the oposite case we use the form

$$\tau_j(g) = -hg(\xi_1)(r^- - q^- p_{j-1}) + hg'(\xi_2)(r^+ - q^+ p_{j+1} + \epsilon(q^- g''(x_{j-1})$$

$$+q^+ g''(x_{j+1}) + q^c g''(x_j)), \;\; x_{j-1} < \xi_1 < x_j < \xi_2 < x_{j+1}, \tag{13}$$

and from Lemma 1 we have

$$|\tau_j(g)| \leq Mh.$$

Thus, the term $\tau_j(g)$ leads to the term in the error estimate which satisfies (10).

For $\tau_j(u)$ and $\tau_j(w)$ using Taylor expansion up to the second derivatives we obtian

$$|\tau_j(u) - \tilde{\tau}_j(u)| \leq Mh^2\epsilon^{-1}x_j^2 u_j \leq$$

$$\leq Mh^2 \exp(-\delta x_j/\sqrt{\epsilon}) \quad \text{when} \quad p'(0) = p'(1) = 0,$$

$$|\tau_j(u) - \tilde{\tau}_j(u)| \leq Mh^2\epsilon^{-1}x_j u_j \leq Mh\exp(-\delta x_j/\sqrt{\epsilon}) \;\; \text{otherwise},$$

$$|\tau_j(w) - \tilde{\tau}_j(w)| \leq Mh^2\epsilon^{-1}(1-x_j)^2 w_j \leq$$

$$\leq Mh^2 \exp(-\delta(1-x_j)/\sqrt{\epsilon}) \quad \text{when} \quad p'(0) = p'(1) = 0,$$

$$|\tau_j(w) - \tilde{\tau}_j(w)| \leq Mh^2\epsilon^{-1}(1-x_j)w_j \leq Mh\exp(-\delta(1-x_j)/\sqrt{\epsilon}) \;\; \text{otherwise}.$$

From (11) we obtain the proof.

Theorem 2. *Let $y(x) \in C^4[0,1]$. Let $v_j, j = 0(1)n+1$, be the approximation to the solution $y(x)$ at the grid points obtainded using scheme (5) or scheme (6). Then*

$$|y(x_j - v_j| \leq M \min(h^2, \epsilon)/\sqrt{\epsilon}. \tag{14}$$

If $p'(0) = p'(1) = 0$ then

$$|y(x_j) - v_j| \leq M \min(h^2, \epsilon) \quad \textit{for scheme (5)}, \tag{15}$$

$$|y(x_j) - v_j| \leq Mh \min(h, \sqrt{\epsilon}) \quad \textit{for scheme (6)}. \;\square \tag{16}$$

Proof. By using Taylor develoments up to the fourth oreder we obtain that

$$\tau_j(y) = T_{2j}y_j'' + T_{3j}y_j''' + R_j(y),$$

where $R_j(y)$ are the remainder terms. In the same way as for Theorem 1 we obtain that

$$|\tau_j(y)| \leq Mh^4/\epsilon \quad \text{for} \quad h^2 \leq \epsilon.$$

When $\epsilon \leq h^2$ we use the form (12) and we have

$$|\tau_j(y) \leq Mh.$$

Since for corresponding matrix the estimate (11) holds, we obtain (14). When $\epsilon \leq h^2$ we use the truncation error in the form (13) in order to obtain the estimate (14). Since

$$q^- = q^+ = 0,\ h^2|r^-|, h^2|r^+| \leq M\epsilon\exp(-\delta h\sqrt{\epsilon}),\ |q^c| \leq Mh/\sqrt{\epsilon}$$

from (11) we obrtain (15). Similarly we obtain the estimate (16) (see [14], [13])).

Corollary 1. *Let Theorem 1 hods and there exists l such that*

$$y^{(k)}(x) = s_k(x)u^{(k)}(x) + p_k(x)w^{(k)}(x)),$$

$$|s_k(x)|, |p_k(x)| \leq M,\ \ x \in [0,1], \quad \textit{for}\ \ l \leq k \leq 4$$

Then

$$|y(x_j) - v_j| \leq \begin{cases} Mh^2\min(h^2,\epsilon)/(\epsilon + h^2) & \textit{for}\ \ p'(1) = p'(0) = 0 \textit{ and } k = 2, \\ Mh^2\min(h^2,\epsilon)/(\epsilon^{3/2} + h^3) & \textit{for } p'(1) \neq p'(0) \textit{ and } k = 1. \ \square \end{cases}$$

Proof. Proof for the second inequality follows from the fact that for $\epsilon \leq h^2$

$$|\tau_j(g)| \leq Mh\exp(-\bar{\delta}x_j/\sqrt{\epsilon}) \leq M\sqrt{\epsilon}\exp(-\delta x_j/\sqrt{\epsilon}),\ j = 2(1)n,$$

(see (13) and Lemma 1) and

$$|\tau_i(g)| \leq M\sqrt{\epsilon},$$

which one can obtain using the integral form of the remainder terms in (13).

Example 2 has the properties wich related to the second estimate of the theorem. The order of the convergence is calculated in respect to h although the scheme converges in respect to ϵ. Because of that we obtain a negative order of the convergence which agree with mentioned estimate, Table 2.

Corollary 2. *Let Theorem 2 holds and there exists ℓsuch that*

$$y^{(k)}(x) = s_k(x)u^{(k)}(x) + p_k(x)w^{(k)}(x)), \tag{17}$$

$$|s_k(x)|, |p_k(x)| \leq M, \quad x \in [0,1],$$

for $l \leq k \leq 4$. Then for the scheme (5) inequalites (14) (when $k = 1$) and (15) (when $k = 2$) holds if we replace M by $M\exp(\delta h/\sqrt{\epsilon})$. For $k = 2$ estimate (16) holds when the right hand side is replaced by $Mmin(h^2, \epsilon)$.

Proof. Using (17) and integral form of the remainder terms in (12) and (13) with the fact that $|r^-|, |r^+| \leq Mexp(-\delta/\sqrt{\epsilon})$ for $\epsilon^2 \leq h$, we obtain the statement.

Remark 1. From Corrolary 1 and Corrolary 2 schow that in some case for a wery small ϵ, the scheme (5) gives bater results than scheme (7). Table 1, Table 3 and Table 4 ilustrate this fact.

4. Numerical eksperiments

In this section we present numerical results using the schemes described in previous section. The tables contain the maximum of differences between exact and approhimate solution at the grid ponts and the numerical validation of the theoretical oreder of convergence in respect to h. The oreder Ord is obtained by

$$Ord = (\log(E_n) - \log(E_{2n}))/\log 2,$$

where $E_n = \max_{1 \leq i \leq n} |v_{i,k} - v_{i,k+1}|$ and $v_{j,k}$ is the approximate solution obtained with the step $h = 1/2^{k+2}$. The stars denote round off error. Example 1. ([2])

Problem: $-\epsilon y" + y + \cos^2(\pi x) + 2\epsilon\pi^2\cos(2\pi x) = 0.\ y(0) = y(1) = 0.$

Solution: $y(x) = (\exp(-x\epsilon) + \exp(-(1-x)/\epsilon))/(1+\exp(-1/\epsilon)) - \cos^2(\pi x).$

Example 2. ([2])

Problem: $-\epsilon y" + (1 + x(1-x))y = f(x), y(0) = y(1) = 0.$

Solution: $y(x) = 1 + (x-1)\exp(-x/\epsilon) - x\exp((x-1)/\epsilon).$

Table 1. Example 1. Scheme (7).

$k \setminus \epsilon$	2^{0}	2^{-8}	2^{-16}	2^{-22}	2^{-26}	2^{-31}	
1	1.5195(-3)	7.2427(-4)	1.9884(-3)	2.8132(-4)	7.1213(-5)	1.7276(-5)	E_n
	—	—	—	—	—	—	Ord
2	9.3253(-5)	5.4240(-5)	8.8890(-4)	1.4397(-4)	8.9713(-6)	E_n	
	4.1137	2.8490	.5582	.5582	.5582	.5582	Ord
3	5.8020(-6)	3.5457(-6)	3.0325(-4)	7.0352(-5)	1.8470(-5)	4.5206(-6)	E_n
	4.0276	3.7243	.9087	.8998	.8998	.8998	Ord
4	3.6222(-7)	2.2406(-7)	5.6865(-5)	3.2913(-5)	9.1107(-6)	2.2572(-6	E_n
	4.0068	3.9319	1.2493	.9755	.9755	.9755	Ord
5	2.2629(-8)	1.4046(-8)	5.5761(-6)	1.4115(-5)	4.4111(-6)	1.2078(-6)	E_n
4.0020	3.9830	2.2631	.9939	.9939	.9939	Ord	
6	1.4555(-9)	8.7848(-10)	3.9840(-7)	4.7626(-6)	2.0558(-6)	5.5183(07)	E_n
	—	—	—	—	—	—	Ord
7	8.9261(-11)	5.4900(-11)	2.5790(-8)	8.8240(-7)	2,6732(-7)	E_n	

Table 2. Example 2. Scheme (7).

$k \setminus \epsilon$	2^{0}	2^{-8}	2^{-16}	2^{-22}	2^{-26}	2^{-31}	
1	2.1259(-6)	2.1359(-3)	5.7948(-5)	90544(-7)	5.6590(-8)	3.3096(-9)	E_n
	—	—	—	—	—	—	Ord
2	1.3296(-7)	2.6668(-4)	1.18538(-6)	1.1586(-7)	6.7760(-9)	E_n	
	3.9956	1.1859	-1.0490	-1.0490	-1.0491	*****	Ord
3	8.3116(-9)	1.7630(-5)	2.4366(-4)	3.7582(-6)	2.3489(-7)	1.3737(-8)	E_n
	3.9989	2.9897	-1.0232	-1.0338	-1.0338	*****	Ord
4	1.4182(-9)	1.1326(-6)	3.5913(-4)	7.5714(-6)	4.7321(-7)	2.7675(-8)	E_n
	3.9997	3.9149	-1.0139	-1.0196	-1.0196	*****	Ord
5	8.8622(-11)	7.1374(-8)	1.4377(-4)	1.5202(-6)	9.5000(-7)	*****	E_n
	4.0000	3.9593	1.3017	-1.0105	-1.0105	*****	Ord
6	5.3832(-12)	4.4688(-9)	2.1985(-5)	3.0990(-5)	1.9036(-6)		E_n
	—	—	—	—	—	—	Ord
7	1.2327 (-11)	2.7945(-10)	1.0991(-6)	4.5509(-5)	3.8115(-6)	*****	E_n

Table 3. Example 1. Scheme (5)

$k \setminus \epsilon$	2^{0}	2^{-8}	2^{-16}	2^{-22}	2^{-26}	
1	4.9987(-2)	2.1793(-2)	3.0120(-4)	4.7062(-6)	2.9414(-7)	E_n
2	1.2215(-2)	6.16480(-3)	3.0119(-4)	4.7062(-6)	2.9414(-7)	E_n
3	3.0366(-3)	1.5906(-3)	2.9474(-4)	4.7062(-6)	2.9414(-7)	E_n
4	7.5809(-4)	4.0080(-4)	2.0964(-4)	4.7062(-6)	2.9414(-7)	E_n
5	1.8945(-4)	1.0040(-4)	8.3119(-5)	4.7061(-6)	2.9414(-7)	E_n
6	4.7360(-5)	2.5112(-4)	2.3894(-5)	4.6051(-6)	2.9414(-7)	E_n
7	1.1840(-5)	6.2789(-6)	6.1973(-6)	3.2751(-6)	2.9413(-7)	E_n

Table 4. Example 2. Scheme (5)

$k \setminus \epsilon$	2^0	2^{-4}	2^{-8}	2^{-12}	2^{-14}	2^{-16}	
1	2.6179(-4)	1.2888(-3)	3.8185(-4)	5.4985(-5)	4.2356(-8)	7.7505(-15)	E_n
2	6.5778(-5)	3.3571(-4)	7.4883(-4)	4.5708(-4)	3.5787(-5)	2.9598(-8)	E_n
3	1.6465(-5)	8.4944(-5)	2.3089(-4)	1.0677(-4)	2.6071(-4)	2.0506(-5)	E_n
4	4.1175(-6)	2.1296(-5)	6.3897(-5)	1.8453(-4)	6.1712(-5)	1.3909(-4)	E_n
5	1.0295(-6)	5.3280(-6)	1.6165(-5)	5.8087(-5)	9.2052(-5)	3.3072(-5)	E_n
6	2.5737(-7)	1.3322(-6)	4.0557(-6)	1.5967(-5)	2.9077(-5)	4.5972(-5)	E_n
7	6.4344(-7)	3.3308(-7)	1.0154(-6)	4.0387(-6)	7.9838(-6)	1.4547(-5)	E_n

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