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NUMERICAL SOLUTION OF SOME DISCRETE ANALOGUES OF BOUNDARY VALUE PROBLEM

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Abstract

Some finite difference analogues for singularly perturbed nonlocal boundary value problem and their numerical solution are considered. We compare two similar discrete analogues based on discretization using special nonequidistant mesh and Hermite scheme. The fourth order convergence uniform in the perturbation parameter is proved. Numerical experiments are presented.

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1. Introduction

In this paper we shall consider the following singularly perturbed boundary value problem:

$$-\epsilon^2 u'' + c(x)u = f(x), \quad x \in I = [0,1], \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$u(1) = \sum_{i=1}^{m} d_i u(s_i) + \delta, \quad 0 < s_1 < s_2 < \cdots < s_m < 1, \tag{3}$$

where $\epsilon \in (0, \epsilon_0), \epsilon_0 \ll 1$, is a small perturbation parameter and δ, $d_i \in \mathbf{R}, i = 1, 2, \ldots, m$.Throughout the paper we shall assume:

$$c, f \in C^4(I), \tag{4}$$

$$0 < \gamma^2 \leq c(x), \quad x \in I, \tag{5}$$

$$\sum_{i=1}^{m} |d_i| < 1. \tag{6}$$

General "boundary value problems" with nonlocal conditions are defined in [8]. Nonlocal conditions which connect the values of the unknown solution at the boundary with values in the interior are defined and studied in [2]. Some problems with integral nonlocal conditions can be met in studying heat transfer problems, [6], [7]. Discretization of these problems leads to the nonlocal conditions of the form (3). The same mathematical models arise in problems of semiconductors, hydrodynamics and some others physical phenomena.

Numerical treatment of problem (1)-(3) was considered in [5] and [13]. In [5] a finite elements method on an equidistant mesh was applied and the second order uniform convergence was obtained. In [13] using results from [12] the fourth order uniform convergence is obtained. The basic idea in the papers mentioned is to consider the problem (1)-(3) as two problems with Dirichlet's boundary conditions which are examined in [3], [4], [9], [12], [15], [16], [23]-[26]. Our aim is to solve (1)-(3) numerically using a finite difference schemes on a special nonequidistant meshes which are dense in the layers, solving only one system of linear equations instead two as in [5] and [13].

The conditions (2), (3), (4), (6) guarantee that problem (1)-(3) has a unique solution $u_\epsilon \in C^6(I)$,which exhibits two boundary layers at the end-points of I. In particular, the following estimates hold, see [24]:

$$| u_\epsilon^{(i)}(x) | \leq \begin{cases} M\{1 + \epsilon^{-i}[\exp(-\gamma x/\epsilon) + \exp(\gamma(x-1)/\epsilon)]\}, & i = 1, 2, 3, 4, \\ M\{\epsilon^{4-i} + \epsilon^{-i}[\exp(-\gamma x/\epsilon) + \exp(\gamma(x-1)/\epsilon)]\}, & i = 5, 6. \end{cases}$$

Here and throughout the paper M denotes any positive constant that may take different values in different formulas, but is always independent of ϵ and of discretization mesh. Because of such behavior of u_ϵ it is necessary to use special methods to solve the problem numerically. We shall use a combination of Hermite and standard central scheme on two special nonequidistant discretization meshes. These meshes will guarantee that the local truncation errors of the schemes will be uniform (by "uniform" we shall always mean "uniform in ϵ"); hence the discretization will be uniformly consistent with the continuous problem. Then the uniform convergence (the convergence of the numerical solution towards the restriction of u_ϵ on the mesh) will follow if we show that our discretization is uniformly stable.

In this paper we shall use a discretization of the same type as in [12] and in [24]. Basically, the Hermite scheme is used, but at some mesh points it is replaced by the standard central scheme. Such a switch is used in order to prove the uniform stability.

We end the paper with some numerical results, which show that theoretical order of convergence is also established numerically.

2. The discrete analogues

1. Let us introduce the discretization mesh $I_s = I_h \cup \{s_1, s_2, \ldots, s_m\}$,where

$$I_h = \{x_i = \lambda(ih) : i = 0, 1, \ldots, n\}, \quad h = \frac{1}{n}, n \in \mathbf{N},$$

with mesh generating function [12], [23]:

$$\lambda(t) = \begin{cases} \mu(t) = \frac{a\epsilon t}{q-t}, & t \in [0, \alpha), \\ \mu'(\alpha)(t-\alpha) + \mu(a), & t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), & t \in [0.5, 1]. \end{cases}$$

Here q is an arbitrary number from (0, 0.5) and $a \in (0, q/\epsilon_0)$. The point α is determined from

$$\mu'(\alpha)(t-\alpha) + \mu(a) = 0.5,$$

which reduces to a quadratic equation and α is

$$\alpha = \frac{q - \sqrt{aq\epsilon(1 - 2q + 2a\epsilon)}}{1 + 2a\epsilon}.$$

It is easy to see that $\lambda(t)$ is a monotone increasing function on I, and we can consider the points s_i as values of $\lambda(\tau_i)$ for some $\tau_i \in (0,1), i =$

$1, 2, \ldots, m$. From now on we denote the points of the mesh I_s as $x_i, i = 0, 1, 2, \ldots, N, n \le N \le n + m$.The other properties of mesh generating function $\lambda(t)$, which are important for our analysis are given in [12], [23], [26].

Let

$$Q = 2(1 + \frac{\sqrt{3}}{3}),$$

$$I'_h = \{x_i \in I_h : q - Qh < (i-1)h < \alpha \quad or \quad 1 - \alpha < (i+1)h < 1 - Qh\}.$$

Let us note that the set I'_h can be empty.

Let $h_i = x_i - x_{i-1}, i = 1, 2, \ldots, N$. Using the Hermite difference scheme from [12] we approximate the differential equation (1) at $x_i \in I_s$by

$$A_i u_{i-1} + B_i u_i + C_i u_{i+1} = D_i.$$

Here

$$A_i = \epsilon^2 a_1(i) + b_1(i) c_{i-1},$$

$$B_i = \epsilon^2 a_0(i) + b_0(i) c_i,$$

$$C_i = \epsilon^2 a_2(i) + b_2(i) c_{i+1},$$

$$D_i = b_1(i) f_{i-1} + b_0(i) f_i + b_2(i) f_{i+1},$$

$$a_1(i) = \frac{-2}{h_i(h_i + h_{i+1})}, \quad a_0(i) = \frac{2}{h_i h_{i+1}}, \quad a_2(i) = \frac{-2}{h_{i+1}(h_i + h_{i+1})},$$

$$b_1(i) = \begin{cases} -a_1(i)(h_i^2 - h_{i+1}^2 + h_i h_{i+1})/12, & if \ x_i \in I_h \backslash I'_h, \\ 0, \quad if \ x_i \in I'_h, \end{cases}$$

$$b_2(i) = \begin{cases} -a_2(i)(h_{i+1}^2 - h_i^2 + h_i h_{i+1})/12, & if \ x_i \in I_h \backslash I'_h, \\ 0, \quad if \ x_i \in I'_h, \end{cases}$$

$$b_0(i) = 1 - b_1(i) - b_2(i),$$

$$c_i = c(x_i), \quad f_i = f(x_i).$$

Using this and conditions (2) and (3) we form a discrete analogue of our problem

$$(7)\quad \begin{bmatrix} 1 & & & & & & \\ A_1 & B_1 & C_1 & & & & \\ & A_2 & B_2 & C_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & A_{N-1} & B_{N-1} & C_{N-1} \\ 0 & \delta_1 & \delta_2 & \cdots & \delta_{N-2} & \delta_{N-1} & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} 0 \\ D_1 \\ D_2 \\ \vdots \\ D_{N-1} \\ \delta \end{bmatrix},$$

where

$$\delta_i = \begin{cases} -d_i, & if \quad x_i \in I'_h, \\ 0, & if \quad x_i \notin I'_h. \end{cases}$$

Theorem 1 *Suppose that conditions (4),(5) and (6) are satisfied. Let* $|c'(x)| \leq L, x \in I$, *and let* u_ϵ *be the solution of (1)-(3) and*

$$u_\epsilon^h = [u_\epsilon(0), u_\epsilon(x_1), \ldots, u_\epsilon(x_{N-1}), u_\epsilon(1)]^\top.$$

If

$$n > \max\left\{\frac{2L}{3\gamma^2(1-2q)K}, \frac{Q}{q}\right\}, \quad 0 < K < 1,$$

then there exists a unique solution $u = [u_0, u_1, \ldots, u_{N-1}, u_N]^\top$ *to (7) and it holds*

$$(8)\qquad \left\|u_\epsilon^h - u\right\|_\infty \leq Mh^4.$$

Proof. As in ([12]) we can prove that the matrix, say **F**, of the system (7) is strictly diagonally dominant and that $\|F^{-1}\|_\infty \leq M$. Let

$$R_h = F(u_\epsilon - u),$$

then we have

$$\|u_\epsilon - u\|_\infty \leq M\, \|R_h\|_\infty.$$

Since $\|R_h\|_\infty \leq Mh^4$,what we prove in the same way as in ([12]), it follows (8). □

2. Let the discretization mesh I_s is defined as in the first case but now with $n \geq 4$, ([24]) and with the following mesh generating function:

$$\lambda(t) = \begin{cases} \mu(t) = \frac{a\epsilon t}{q-t}, \quad t \in [0,\alpha], \\ \pi(t) = \omega(t-\alpha)^3 + 0.5\mu''(\alpha)(t-\alpha)^2 + \\ \qquad\qquad \mu'(\alpha)(t-\alpha) + \mu(a), t \in [\alpha, 0.5], \\ 1 - \lambda(1-t), \quad t \in [0.5, 1]. \end{cases}$$

Here q is an arbitrary parameter from $(\epsilon_0^{1/3}, 0.5)$ and $\alpha = q - \epsilon^{1/3} > 0$, where we assume that $8\epsilon_0 < 1$. The coefficient ω is determined from $\pi(0.5) = 0.5$:

$$\omega = (0.5-\alpha)^{-3}\left\{0.5 - a\left[q(0.5-\alpha)^2 + q(0.5-\alpha)\epsilon^{1/3} + \alpha\epsilon^{2/3}\right]\right\},$$

and a is chosen so that $\omega \geq 0$. The mesh I_h' is defined

$$I_h' = \{x_i \in I_h : b_1(i) \geq 0, \quad b_2(i) \geq 0 \quad and \quad \rho_i \leq 1\}.$$

Now, we can write our new discrete analogue in the form (7), where we calculate corresponding coefficients using new meshes I_h, I_h' and I_s instead of the old one.

The proof of the following theorem follows directly from ([24]).

Theorem 2 *Suppose that conditions (4),(5) and (6) are satisfied. Let* $c(x) \leq \Gamma, x \in I$, *and let* u_ϵ *be the solution of (1)-(3) and*

$$u_\epsilon^h = [u_\epsilon(0), u_\epsilon(x_1), \ldots, u_\epsilon(x_{N-1}), u_\epsilon(1)]^\top.$$

If ϵ_0 *is sufficiently small. Then for* $\epsilon \in (0, \epsilon_0]$ *there exists a unique solution* $[u_0, u_1, \ldots, u_{N-1}, u_N]^\top$ *to (7) and it holds*

$$\left\|u_\epsilon^h - u\right\|_\infty \leq Mh^4. \tag{9}$$

3. Numerical results

In order to compare two described analogue we shall consider the following test problem:

$$-\epsilon^2 u'' + u = 1, \quad x \in I = [0,1],$$

$$u(0) = 0,$$

$$u(1) = \sum_{i=1}^{m} d_i u(s_i) + \delta, \quad 0 < s_1 < s_2 < \cdots < s_m < 1,$$

where for given $m, s_i, d_i, i = 1, 2, \ldots, 4$,

i	1	2	3	4
s_i	0.10	0.30	0.70	0.9999
d_i	0.11	0.23	0.55	0.10

δ is so chosen that $u(1) = 1$. In this case the exact solution $u_\epsilon(x)$ is known, ([12]):

$$u_\epsilon(x) = 1 - ch(x/\epsilon) - sh(x/\epsilon)\frac{1 - ch(1/\epsilon)}{sh(1/\epsilon)}.$$

We denote by E_n the maximum of $\left|u_\epsilon^h(x_i) - u_i\right|$, $i = 1, 2, \ldots, N-1$, i.e.

$$\left\|u_\epsilon^h - u\right\|_\infty .$$

Also, we define in the usual way the order of convergence Ord for two successive values of n with respective errors E_n and E_{2n} :

$$Ord_n = \frac{\log E_n - \log E_{2n}}{\log 2}.$$

We expect that $Ord = 4$ in both cases.

We shall always use the mesh with $a = 1$ and $q = 0.48$. By changing these parameters, it is possible to change the percentage of the mesh points lying in the layers, ([15]), ([12]), ([23]), ([26]). Here is $\gamma = \Gamma = 1$ and by the first analogue it should be $n > \frac{Q}{q} = 3.1547\ldots.$ and by the second $n \geq 4$. In the tables we have $\epsilon = 2^{-k}$.

Table1. Ord_n

n\k	4	5	6	7	8	9	10-48
8	1.495	2.501	2.623	2.642	2.651	2.657	2.660
16	5.685	2.813	1.708	1.706	1.704	1.705	1.705
32	3.947	3.651	4.245	4.248	4.250	4.250	4.250
64	3.977	3.948	4.063	4.063	4.063	4.063	4.062
128	3.971	3.995	3.993	4.011	4.011	4.011	4.011
256	3.992	3.988	4.023	4.006	4.006	4.006	4.006
512	4.001	4.010	3.938	4.001	4.001	4.001	4.001

Table2. Ord_n

n\k	4	6	8	10	12	14	16	18	20-48
8	2.984	2.948	1.760	1.954	2.132	2.451	2.649	2.649	1.833
16	3.389	2.161	1.948	1.070	3.494	2.803	4.073	4.073	3.328
32	2.519	5.288	2.123	1.949	3.399	5.483	3.683	4.169	3.747
64	3.834	4.187	5.781	1.771	0.663	1.770	1.699	4.081	3.864
128	.0743	1.451	4.115	2.516	1.518	0.787	1.648	4.124	4.011
256	1.058	2.448	4.043	6.575	2.352	1.157	2.804	4.062	4.006
512	3.925	4.015	4.007	4.017	6.194	3.182	3.088	4.029	4.001

These results confirm the fourth order of uniform convergence, obtained theoretically. The results in Table 2 are worse for $\epsilon < 2^{-20}$ than the corresponding results from Table 1. For other values of ϵ the results are the same. The number of the mesh points lying in $[0,\epsilon\ ]$, is the same in both cases.

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