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ON COINCIDENCE POINT THEOREM FOR MULTIVALUED MAPPINGS IN PROBABILISTIC METRIC SPACES

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Abstract

A coincidence point theorem for multivalued mappings in probabilistic metric spaces is proved, which is a generalization of the fixed point theorem proved by V.Radu [3].

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1. Introduction

T. L. Hicks proved in [2] a fixed point theorem for the so-called $C-$ contraction in probabilistic metric spaces $(S, \mathcal{F}, \min)$. A mapping $f : S \to S$ is a $C-$ contraction if there is a $k \in (0,1)$ such that for every $p, q \in S$ and $x > 0$

$$F_{p,q}(x) > 1 - x \Rightarrow F_{fp,fq}(kx) > 1 - kx.$$

V. Radu proved [3] that $f : S \to S$ is a C - contraction on S, where $(S, \mathcal{F}, T)$ is a complete Menger space with $T \geq T_m$, $T_m(a,b) = \max\{a +$

$b-1,0\}$ $(a,b\in[0,1])$, if and only if f is a metric contraction on the metric space (S,β),

$$\beta(p,q)=\inf\{h;\ F_{p,q}(h^+)>1-h\}.$$

If $\sup\limits_{x<1} T(x,x)=1$. V. Radu proved that a $C-$ contraction $f:S\to S$ has a fixed point. In this paper, a generalization of this result for multivalued mappings will be proved.

2. Preliminaries

Let $(S,\mathcal{F},t)$ be a Menger space, $\emptyset\neq M\subset S$, $f:M\to M$ and $A:M\to 2^M$ (the family of nonempty subsets of M). The mapping A is $f-$ strongly demicompact if for every sequence $\{x_n\}_{n\in\mathbf{N}}$ from M, such that $\lim\limits_{n\to\infty} F_{fx_n,y_n}(\epsilon)=1$, for some sequence $\{y_n\}_{n\in\mathbf{N}}$, $y_n\in Ax_n$, $n\in\mathbf{N}$ and every $\epsilon>0$, there exists a convergent subsequence $\{fx_{n_k}\}_{k\in\mathbf{N}}$.

A mapping $A:M\to 2^M$ is weakly commuting with $f:M\to M$ if for every $x\in M$

$$f(Ax)\subset A(fx).$$

A $t-$norm T is of the $h-$type if the family $\{T_p(x)\}_{p\in\mathbf{N}}$ is equicontinuous at the point $x=1$, $T_1(x)=T(x,x)$, $T_p(x)=T(T_{p-1}(x),x)$, $p\geq 2$, $x\in[0,1]$.

A nontrivial example of such a $T-$norm can be found in [1].

Let

$$\mathcal{M}=\{\psi;\psi:\mathbf{R}_+\to\mathbf{R}_+,\psi \text{ is nondecreasing and } \sum_{n=1}^{\infty}\psi^n(u)<\infty,$$
$$\text{for every }\ u\in\mathbf{R}_+\}.$$

By $2_c^{f(M)}$ we shall denote the family of all nonempty, closed subsets of $f(M)$.

3. A fixed point theorem

Theorem 1. *Let $(S,\mathcal{F},T)$ be a complete Menger space,* $\sup_{x<1}T(x,x)=$

1, M a nonempty and closed subset of S, $f : M \to M$ a continuous mapping, $A, B : M \to 2_c^{f(M)}$ and $\psi \in \mathcal{M}$, so that the following implication holds for every $u, v \in M$ and every $\epsilon > 0$:

$$F_{fu,fv}(\epsilon) > 1 - \epsilon \quad \Rightarrow \quad \begin{array}{l} \text{for every } p \in Au \text{ there exists } q \in Bv \\ \text{such that } F_{p,q}(\psi(\epsilon)) > 1 - \psi(\epsilon) \text{ and} \\ \text{for every } p' \in Bv \text{ there exists } q' \in Au \\ \text{such that } F_{p',q'}(\psi(\epsilon)) > 1 - \psi(\epsilon). \end{array}$$

If A and B are weakly commuting with f and (a) or (b) are satisfied, then there exists $x \in M$ such that $fx \in Ax \cap Bx$, where

a) A or B are $f-$strongly demicompact.

b) $t-$norm T is of the $h-$type.

Proof. Let $x_0 \in M$ and $x_1 \in M$ be such that $fx_1 \in Ax_0$. If $s > 1$, then $F_{fx_0,fx_1}(s) > 1 - s$ and so there exists $x_2 \in M$ such that $F_{fx_1,fx_2}(\psi(s)) > 1 - \psi(s)$ and $fx_2 \in Bx_1$. Continuing in this way we obtain a sequence $\{x_n\}_{n\in \mathbf{N}}$ in M such that for every $n \in \mathbf{N}$

(i) $fx_{2n+1} \in Ax_{2n}, fx_{2n+2} \in Bx_{2n+1}$

(ii) $F_{fx_n,fx_{n+1}}(\psi^n(s)) > 1 - \psi^n(s)$.

Since $\lim\limits_{n\to\infty} \psi^n(s) = 0$, from (ii) it is easy to prove that for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $n_1(\epsilon,\lambda) \in \mathbf{N}$ such that for every $n \geq n_1(\epsilon,\lambda)$ $F_{fx_n,fx_{n+1}}(\epsilon) > 1 - \lambda$. This means that for every $\epsilon > 0$

$$(1) \qquad \lim_{n\to\infty} F_{fx_n,fx_{n+1}}(\epsilon) = 1.$$

If we suppose that A is $f-$ strongly demicompact, using

$$\lim_{n\to\infty} F_{fx_{2n},fx_{2n+1}}(\epsilon) = 1 \text{ and } fx_{2n+1} \in Ax_{2n} \ (n \in \mathbf{N}),$$

we conclude that there exists a convergent subsequence $\{fx_{2n_k}\}_{k\in\mathbf{N}}$ of the sequence $\{fx_{2n}\}_{n\in\mathbf{N}}$.

We shall prove that if T is of the $h-$type, the sequence $\{fx_n\}_{n\in\mathbf{N}}$ is convergent.

Let $\epsilon > 0$ and $\lambda \in (0,1)$. Since the series $\sum_{n=1}^{\infty} \psi^n(s)$ is convergent, there exists $n'(\epsilon, s) \in \mathbf{N}$ such that $\sum_{n \geq n'(\epsilon,s)} \psi^n(s) < \epsilon$. Then, for every $n \geq n'(\epsilon, s)$

$$F_{fx_{n+p+1}, fx_n}(\epsilon) \geq \underbrace{T(T(...T(}_{p-\ times}(F_{fx_{n+p+1}, fx_{n+p}}(\psi^{n+p}(s)),$$

$$F_{fx_{n+p}, fx_{n+p-1}}(\psi^{n+p-1}(s)), ..., F_{fx_{n+1}, fx_n}(\psi^n(s))).$$

If $n \geq n''(\epsilon, s) = \max\{n(s),\ n'(\epsilon, s)\}$, where $\psi^n(s) < 1$, for $n \geq n(s)$, then

$$F_{fx_{n+p}, fx_n}(\epsilon) \geq \underbrace{T(T(...T(}_{p-\ times}1 - \psi^{n+p}(s),\ 1 - \psi^{n+p-1}(s)), ..., 1 - \psi^n(s)).$$

Hence

$$F_{fx_{n+p+1}, fx_n}(\epsilon) \geq T_p(1 - \psi^n(s)) \tag{2}$$

for every $n \geq n''(\epsilon, s)$ and every $p \in \mathbf{N}$. Since the family $\{T_p(x)\}_{p \in \mathbf{N}}$ is equicontinuous at the point $x = 1$, for every $\lambda \in (0,1)$ there exists $\delta(\lambda) \in (0,1)$ such that $T_p(1 - \delta(\lambda)) > 1 - \lambda$, for every $p \in \mathbf{N}$. If $n'''(s, \lambda) \in \mathbf{N}$ is such that $\psi^n(s) < \delta(\lambda)$, for every $n \geq n'''(s, \lambda)$ it follows from (2) that

$$F_{fx_{n+p+1}, fx_n}(\epsilon) > 1 - \lambda$$

for every $n \geq \max\{n''(\epsilon, s),\ n'''(s, \lambda)\}$.

Since S is complete and M is closed we conclude that in both cases (a) and (b) there exists $x = \lim_{k \to \infty} fx_{2n_k} \in M$. From (1) it follows that $x = \lim_{k \to \infty} fx_{2n_k+1}$.

We shall prove that $fx \in Ax \cap Bx$. Since Ax and Bx are closed it remains to be proved that $fx \in \overline{Ax} \cap \overline{Bx}$ i.e. that for every $\epsilon > 0$ and $\lambda \in (0,1)$ there exists $q(\epsilon, \lambda) \in Ax$ and $r(\epsilon, \lambda) \in Bx$ such that

$$q(\epsilon, \lambda) \in \mathcal{N}_{fx}(\epsilon, \lambda),\ r(\epsilon, \lambda) \in \mathcal{N}_{fx}(\epsilon, \lambda). \tag{3}$$

Since $\sup_{x<1} T(x, x) = 1$ it is easy to see that there exists $\delta(\lambda) \in (0,1)$ such that

$$T(1 - \delta(\lambda), T(1 - \delta(\lambda), 1 - \delta(\lambda))) > 1 - \lambda.$$

From the continuity of f and $x = \lim_{k\to\infty} fx_{2n_k}$ it follows that there exists $k_1 \in \mathbf{N}$ such that

(4) $$F_{fx,ffx_{2n_k}}\left(\frac{\epsilon}{3}\right) > 1 - \delta(\lambda), \quad \text{for every} \quad k \ge k_1.$$

From (1) it follows that there exists $k_2 \in \mathbf{N}$ such that

(5) $$F_{ffx_{2n_k},ffx_{2n_k+1}}\left(\frac{\epsilon}{3}\right) > 1 - \delta(\lambda), \quad \text{for every} \quad k \ge k_2.$$

Let $t_0 \in \mathbf{R}_+$ be such that $\psi(t_0) < \min\{\frac{\epsilon}{3}, \delta(\lambda)\}$ and $k_3 \in \mathbf{N}$ such that

(6) $$f_{fx,ffx_{2n_k}}(t_0) > 1 - t_0, \quad \text{for every} \quad k \ge k_3.$$

Since $fx_{2n_k+1} \in Ax_{2n_k}$ $(k \in \mathbf{N})$ and A weakly commutes with f it follows that

(7) $$ffx_{2n_k+1} \in f(Ax_{2n_k}) \subset Afx_{2n_k} \ (k \in \mathbf{N}).$$

Using (6) and (7) we obtain that there exists $r(\epsilon, \lambda) \in Bx$ such that for $k \ge k_3$

$$F_{ffx_{2n_k+1,r(\epsilon,\lambda)}}(\psi(t_0)) > 1 - \psi(t_0)$$

which implies that

$$F_{ffx_{2n_k+1,r(\epsilon,\lambda)}}\left(\frac{\epsilon}{3}\right) \ge F_{ffx_{2n_k+1,r(\epsilon,\lambda)}}(\psi(t_0)) >$$

(8) $$> 1 - \psi(t_0) > 1 - \delta(\lambda), \quad \text{for every} \quad k \ge k_3.$$

Hence, for $k \ge \max\{k_1, k_2, k_3\}$, using (4), (5) and (8) we obtain that

$$F_{fx,r(\epsilon,\lambda)}(\epsilon) \ge T(F_{fx,ffx_{2n_k}}\left(\frac{\epsilon}{3}\right), T(F_{ffx_{2n_k},ffx_{2n_k+1}}\left(\frac{\epsilon}{3}\right),$$

$$F_{ffx_{2n_k+1},r(\epsilon,\lambda)}\left(\frac{\epsilon}{3}\right))) > 1 - \lambda.$$

This means that $r(\epsilon, \lambda) \in \mathcal{N}_{fx}(\epsilon, \lambda)$. From (1) and

$$x = \lim_{k\to\infty} fx_{2n_k} = \lim_{k\to\infty} fx_{2n_k+1} = \lim_{k\to\infty} fx_{2n_k+2}$$

we obtain that there exists $k_1', k_2', k_3' \in \mathbf{N}$ such that

$$(9) \qquad F_{fx, ffx_{2n_k+1}}\left(\frac{\epsilon}{3}\right) > 1 - \delta(\lambda), \quad \text{for every } \; k \geq k_1'$$

$$(10) \qquad F_{ffx_{2n_k+1}, ffx_{2n_k+2}}\left(\frac{\epsilon}{3}\right) > 1 - \delta(\lambda), \quad \text{for every } \; k \geq k_2'$$

$$(11) \qquad F_{fx, ffx_{2n_k+1}}(t_0) > 1 - t_0, \quad \text{for every } \; k \geq k_3'$$

Since $fx_{2n_k+2} \in Bx_{2n_k+1}$ $(k \in \mathbf{N})$ and B weakly commutes with f it follows that $ffx_{2n_k+2} \in f(Bx_{2n_k+1}) \subset B(fx_{2n_k+1})$. Hence, there exists $q(\epsilon, \lambda) \in Ax$ so that for every $k \geq k_3'$

$$F_{ffx_{2n_k+2}, q(\epsilon,\lambda)}(\psi(t_0)) > 1 - \psi(t_0)$$

which implies for $k \geq \max\{k_1', k_2', k_3'\}$

$$F_{fx, q(\epsilon,\lambda)}(\epsilon) \geq T(F_{fx, ffx_{2n_k+1}}\left(\frac{\epsilon}{3}\right),$$
$$T(F_{ffx_{2n_k+1}, ffx_{2n_k+2}}\left(\frac{\epsilon}{3}\right),$$
$$F_{ffx_{2n_k+2}, q(\epsilon,\lambda)}\left(\frac{\epsilon}{3}\right))) > 1 - \lambda.$$

Example. Let (M, d) be a separable metric space, (Ω, Σ, P) a probability space and S the space of all classes of measurable mappings from Ω into M. Then $(S, \mathcal{F}, T_m)$ is a Menger space where

$$F_{X,Y}(u) = P(\{\omega;\ \omega \in \Omega,\ d(X(\omega), Y(\omega)) < u\})\ (u \in \mathbf{R}), X, Y \in S.$$

The Ky Fan metric in S is

$$d(X, Y) = \sup\{u;\ F_{X,Y}(u) < 1 - u, \;\ u > 0\}$$

and the (ϵ, λ) topology and the topology induced by d coincide.

Let $f : S \to S$ be a continuous mapping and $A, B : S \to 2_c^{f(M)}$ such that

$$D(AX, BY) \leq \psi(d(fX, fY)), \;\ X, Y \in S,$$

where ψ is a strictly increasing mapping from $\mathbf{R}_+$ into $\mathbf{R}_+$.

If $F_{fX,fY}(u) > 1-u$ then $d(fX, fY) < u$ and since ψ is strictly increasing we have that $\psi(d(fX, fY)) < \psi(u)$. Hence

$$\sup_{U\in AX} \inf_{V\in BY} d(U,V) < \psi(u), \quad \sup_{V\in BY} \inf_{U\in AX} d(U,V) < \psi(u),$$

which implies that for every $U \in AX$ there exists $V \in BY$ such that $d(U,V) < \psi(u)$ and that for every $V' \in BY$ there exists $U' \in AX$ such that $d(U', V') < \psi(u)$. So

$$F_{U,V}(\psi(u)) > 1 - \psi(u) \quad \text{and} \quad F_{U',V'}(\psi(u)) > 1 - \psi(u).$$

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