Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 1 (1995), 9-18

Review of Research
Faculty of Science
Mathematics Series

FIXED POINT THEOREM IN CONVEX METRIC SPACE

B. K. Sharma, C. L. Dewangan
School of Studies in Mathematics
Pt. Ravishankar Shukla University
Raipur-492010 India

Abstract

Fixed point theorem for convex metric space is proved under more generalized conditions.

AMS Mathematics Subject Classification (1991): 47H10, 54H25
Key words and phrases: convex metric space, fixed points

1. Introduction

Takahashi [5] has introduced the definition of convexity in metric space and generalized some fixed point theorems previously proved for the Banach space. Subsequently, Mochado [3], Tallman [6], Naimpally and Singh [4], Guay and Singh [1], Hadžić and Gajić [2] were among others who obtained results in this setting. This paper is a continuation of the investigation in the same setting.

2. Preliminaries

To prove our result we need the following definitions:

Definition 1. *Let X be a metric space and I be the closed unit interval. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for all $x, y \in X$, $\lambda \in I$,*

$$d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y), \text{ for all } u \in X.$$

The metric space (X, d), together with a convex structure is called the Takahashi convex metric space.

Any subset of a Banach space is a Takahashi convex metric space with

$$W(x,y,\lambda) = \lambda x + (1-\lambda).$$

Definition 2. *Let X be a convex metric space. A nonempty subset K of X is said to be convex if and only if $W(x,y,\lambda) \in K$ whenever $x, y \in K$, $\lambda \in I$.*

Takahashi [5] has shown that the open and closed balls are convex and that an arbitrary intersection of convex sets is also convex.

For an arbitrary $A \subset X$, let

$$\tilde{W}(A) = \{W(x,y,\lambda) : \ x, y \in A, \ \lambda \in I\}. \tag{1}$$

It is easy to see that

$\tilde{W} : P(X) \to P(X)$ is a mapping with the properties:

(i) $A \subset \tilde{W}(A)$, for $A \subset X$,

(ii) $A \subset B \Rightarrow \tilde{W}(A) \subset \tilde{W}(B)$, for $A, B \in P(X)$,

(iii) $\tilde{W}(A \cap B) \subset W(A) \cap \tilde{W}(B)$, for any $A, B \in P(X)$.

Using this notation we can see that K is convex iff $\tilde{W}(K) \subset K$.

Definition 3. *A convex metric space X will be said to have Property (C) iff every bounded decreasing set of nonempty closed convex subset of X has nonempty intersection.*

Remark 1. Every weakly compact convex subset of a Banach space has Property (C).

Definition 4. *Let X be a convex metric space and A be a nonempty closed, convex bounded set in X. For $x \in X$, let us set*

$$r_x(A) = \sup_{y \in A} d(x, y),$$

and

$$r(A) = \inf_{x \in A} r(A).$$

We thus define $A_C = \{x \in A : \ r_x(A) = r(A)\}$ to be the centre of A.

We denote the diameter of a subset A of X by

$$\delta(A) = \sup\{d(x, y) : \ x, y \in A\}.$$

Definition 5. *A point $x \in A$ is a diametral point of A iff*

$$\sup_{y \in A} d(x, y) = \delta(A).$$

Definition 6. *A convex metric space X is said to have normal structure iff for each closed bounded, convex subset A of X, containing at least two points, there exists $x \in A$, which is not a diametral point of A.*

Remark 2. Any compact convex metric space has a normal structure.

Definition 7. *A convex hull of the set A $(A \subset X)$ is the intersection of all convex sets in X containing A, an is denoted by $\operatorname{conv} A$.*

It is obvious that if A is a convex set, then

$$\tilde{W}^n(A) = \tilde{W}(\tilde{W}(\tilde{W}(A)\ldots)) \subset A \text{ for any } n \in \mathbf{N}.$$

If we set

$$A_n = \tilde{W}^n(A), \ (A \subset X),$$

then the sequence $\{A_n\}_{n \in \mathbf{N}}$ will be increasing and $\limsup A_n$ exists, and $\limsup A_n = \liminf A_n = \lim A_n = \bigcup_{n=1}^{\infty} A_n$.

Also, we need the following propositions:

Proposition 1. *Let X be a convex metric space. Then*

$$(2) \qquad conv\, A = \lim A_n = \bigcup_{n=1}^{\infty} A_n, \ (A \subset X)$$

Proof. If $x, y \in \bigcup_{n=1}^{\infty} A_n$, then there exists a positive integer n_0 (say) such that $x, y \in A_{n_0}$. So, for $\lambda \in I$,

$$W(x, y, \lambda) \in A_{n_0+1} \subset \bigcup_{n=1}^{\infty} A_n.$$

Thus, $\bigcup_{n=1}^{\infty} A_n$ is convex and contains A.

Further, for any convex set C containing A,

$$\tilde{W}^n(A) \subset C \text{ for every } n \in \mathbf{N}$$

i. e.

$$\bigcup_{n=1}^{\infty} A_n \subset C.$$

So,

$$\bigcup_{n=1}^{\infty} A_n = \operatorname{conv} A.$$

In the remaining part of this paper (X, d) will denote a convex metric space.

Proposition 2. *For any subset A of* (X, d)

$$\delta(conv\, A) = \delta(A).$$

Proof. Since $A \subset \operatorname{conv} A$ then $\delta(A) \leq \delta(\operatorname{conv} A)$. Now let x and y be in $\operatorname{conv} A$. If $x, y \in A$, then it is obvious that $d(x, y) \leq \delta(A)$. So, let one of them, for instance x, be in $\operatorname{conv} A$, and let the other one be from A. Since $x \in \operatorname{conv} A$, there exists $n_0 \in \mathbf{N}$ such that $x \in \tilde{W}^{n_0}(A)$. But, it means that there exist $x_1, x_2 \in \tilde{W}^{n_0-1}(A)$, $\lambda_1 \in [0, 1]$, so that $x = W(x_1, x_2, \lambda_1)$ and then:

$$d(x, y) = d(W(x_1, x_2, \lambda_1), y) \leq \lambda_1 d(x_1, y) + (1 - \lambda_1) d(x_2, y).$$

By induction it can be seen that there exists the subset

$$\{x_i\}_{i\in I} \subset A \ (I - \text{finite set})$$

and

$$\{\alpha_i\}_{i\in I},\ \alpha_i \geq 0,\ \sum_{i\in I} \alpha_i = 1,$$

such that

$$d(x,y) \leq \sum_{i\in I} \alpha_i d(\tilde{x}_i, y).$$

Since

$$d(\tilde{x}_i, y) \leq \delta(A) \text{ for } i \in I,$$

we can prove that

(3) $$d(x,y) \leq \delta(A).$$

Similarly, it can be seen that the above is valid even in the case for $y \in \operatorname{conv} A$.

3. Main result

Now we prove the following:

Theorem 1. *Let (X,d) be a metric space with continuous convex structure and let K be a closed convex bounded subset of (X,d) with normal structure and Property (C).*

If $A : K \to K$ is a continuous mapping such that for $x, y \in K$,

(4) $$\begin{aligned} d(Ax,Ay) \leq \max\{&d(x,y), d(x,Ax), d(y,Ay), d(x,Ay), d(y,Ax) \\ &d(x,A^2x), d(y,A^2y), d(Ax,A^2x), d(Ay,A^2y)\} \end{aligned}$$

then A has a fixed point.

Proof. Let $\mathcal{F}$ be a family of non-empty closed convex subsets $F \subset K$ so that $A(F) \subset F$, then F is non-empty since $K \in F$. We partially order F by inclusion, and let $S = \{F_i\}_{i\in\Delta}$ be the decreasing chain in F. Then by Property (C) we have that

$$F_0 = \bigcap_{i\in I} F_i \neq \emptyset.$$

So,

$$F_0 \in F.$$

Therefore, any chain in F has a greatest lower bound, and by Zorn's Lemma there is a minimal member $\mathcal{F}$ in F. We claim that F is a singleton set. If not, then, as shown by Takahashi [5], the centre of D, denoted by F_C, is a non-empty proper closed convex subset of F. Now, it is easy to see that

$$\delta(F_C) \leq r(F) \leq \delta(F).$$

Now, let us define a sequence $F_0 = F_c$ and

$$F_{k+1} = \operatorname{conv}(F_k \cup A(F_k)), \; k = 0, 1, \ldots.$$

Clearly, $F_k \subset F_{k+1}$, $(k = 0, 1, \ldots)$. Thus we shall prove by induction that

$$\delta_k = \delta(F_k) \leq r(F) = r, \text{ for any } k \in \mathbf{N}. \tag{5}$$

For $k = 0$ (5) is valid. Suppose that it is valid for $k = 0, 1 \ldots, m$, then we show that it is also valid for $k = m + 1$.

By definition of $\delta(F)$ for any sequence $\{\varepsilon_n\}, \varepsilon_n > 0$ $(n \in \mathbf{N})$, $\lim_{n\to\infty} \varepsilon_n = 0$, there exist $\tilde{x}_n, \tilde{y}_n \in F_{m+1}$, so that

$$\delta_{m+1} - \varepsilon_n \leq d(\tilde{x}_n, \tilde{y}_n).$$

Then, by Proposition 2 we have three cases:

(i) $\tilde{x}_n, \tilde{y}_n \in F_m$ $(n = 1, 2, \ldots)$

(ii) $\tilde{x}_n = x_n$ $\tilde{y}_n = Ay_n$ $(x_n, y_n \in F_m,\ n = 0, 1, \ldots)$

(ii) $\tilde{x}_n = Ax_n$ $\tilde{y}_n = Ay_n$ $(x_n, y_n \in F_m,\ n = 0, 1, \ldots)$

Considering the first case it is clear that $\delta_{m+1} \leq r$. So, let us see the second one. For any $x \in F_0$ thus we have

$$d(x, Ax) \leq r. \tag{6}$$

We assume that (6) is valid for $x \in F_k$ $(k = 0, 1, \ldots, m - 1)$ and prove that it is valid for $k = m$.

For any $x \in F_m$, by Proposition 1, $x \in \tilde{W}^{n_0}(F_{m-1} \cup A(F_{m-1}))$ for some $n_0 \in \mathbf{N}$. Then

$$d(x, Ax) \leq \sum_{j \in I_1} \gamma_j d(x_j, Ax) + \sum_{j \in I_2} \gamma_j d(Ax_j, Ax), \tag{7}$$

for $x_j \in F_{m-1}$, $j \in I = I_1 \cup I_2$, (I-finite set), $I_1 \cap I_2 = \emptyset$ and $\sum_{j \in I} \gamma_j = 1$, $\gamma_j \geq 0$ for $j \in I$. In (7) is sufficient to look only for the case when $\sum_{j \in I} \gamma_j \neq 0$.

Further, we have

$$\begin{aligned} d(x, Ax) \quad \leq \quad & \sum_{j \in I_1} \gamma_j d(x_j, Ax) + \sum_{j \in I_2^{(1)}} \gamma_j d(Ax_j, x) \\ & \sum_{j \in I_2^{(2)}} \gamma_j d(x_j, Ax_j) + \sum_{j \in I_2^{(3)}} \gamma_j d(x_j, Ax) \\ & \sum_{j \in I_2^{(4)}} \gamma_j d(x_j, Ax) + \sum_{j \in I_2^{(5)}} \gamma_j d(x, Ax_j) \\ & \sum_{j \in I_2^{(6)}} \gamma_j d(x_j, A^2x_j) + \sum_{j \in I_2^{(7)}} \gamma_j d(x, A^2x) \\ & \sum_{j \in I_2^{8}} \gamma_j d(Ax_j, A^2x_j) + \sum_{j \in I_2^{(9)}} \gamma_j d(Ax_j, A^2x) \end{aligned}$$

where we suppose
for $I \in I_2^{(1)}$ that $d(Ax_j, Ax) \leq d(x_j, x)$
for $I \in I_2^{(2)}$ that $d(Ax_j, Ax) \leq d(x_j, Ax_j)$
for $I \in I_2^{(3)}$ that $d(Ax_j, Ax) \leq d(x, Ax)$
for $I \in I_2^{(4)}$ that $d(Ax_j, Ax) \leq d(x_j, Ax)$
for $I \in I_2^{(5)}$ that $d(Ax_j, Ax) \leq d(x, Ax_j)$
for $I \in I_2^{(6)}$ that $d(Ax_j, Ax) \leq d(x_j, A^2x_j)$
for $I \in I_2^{(7)}$ that $d(Ax_j, Ax) \leq d(x, A^2x)$
for $I \in I_2^{(8)}$ that $d(Ax_j, Ax) \leq d(Ax_j, A^2x_j)$
for $I \in I_2^{(9)}$ that $d(Ax_j, Ax) \leq d(Ax, A^2x)$.

Now, using the hypothesis, one can see that

$$d(x, Ax) \quad \leq \quad \sum_{j \in I_1} \gamma_j d(x_j, Ax) + r \sum_{j \in I_2^{(1)}} \gamma_j$$

$$
\begin{aligned}
&+ \sum_{j\in I_2^{(2)}} \gamma_j d(x_j, Ax_j) + \sum_{j\in I_2^{(3)}} \gamma_j d(x_j, Ax) \\
&+ \sum_{j\in I_2^{(4)}} \gamma_j d(x_j, Ax) + \sum_{j\in I_2^{(5)}} \gamma_j d(x, Ax_j) \\
&+ \sum_{j\in I_2^{(6)}} \gamma_j d(x_j, A^2x_j) + \sum_{j\in I_2^{(7)}} \gamma_j d(x, A^2x) \\
&+ \sum_{j\in I_2^{8}} \gamma_j d(Ax_j, A^2x_j) + \sum_{j\in I_2^{(9)}} \gamma_j d(Ax_j, A^2x)
\end{aligned}
$$

Since by induction, similarly, we have

$$
\begin{aligned}
d(x, Ax) \quad \leq \quad & \sum_{k\in J_j^{(1)}} \beta_k d(\hat{x}k, x_j) \\
&+ \sum_{k\in J_j^{(2)}} \beta_k d(\hat{x}k, A\hat{x}k) + \sum_{k\in J_j^{(3)}} \beta_k d(x_j, Ax_j) \\
&+ \sum_{k\in J_j^{(4)}} \beta_k d(\hat{x}k, Ax_j) + \sum_{k\in J_j^{(5)}} \beta_k d(x_j, A\hat{x}k) \\
&+ \sum_{j\in I_j^{(6)}} \beta_k d(\hat{x}k, A^2\hat{x}k) + \sum_{k\in J_j^{(7)}} \beta_k d(x_j, A^2x_j) \\
&+ \sum_{j\in I_j^{(8)}} \beta_k d(A\hat{x}k, A^2\hat{x}k) + \sum_{k\in J_j^{(9)}} \beta_k d(Ax_j, A^2x),
\end{aligned}
$$

for $\hat{x}k \in F_{m-1}$ $(k \in J_i = \bigcup_{i=1}^{9} J_i^{(i)}$, $\sum_{k\in J_j} \beta_k = 1$ and $B_k \geq 0$, $k \in J_j$, $\sum_{k\in J_j^{(1)}} \beta_k \neq 0)$. Therefore

$$d(x, Ax_j) \leq r$$

and

$$
\begin{aligned}
d(x, Ax) \quad \leq \quad & \sum_{j\in I_1} \gamma_j d(x_j, Ax) + r\Big(\sum_{j\in I_2^{(1)}} + \sum_{j\in I_2^{(2)}} + \sum_{j\in I_2^{(5)}}\Big)\gamma_j \\
&+ \sum_{j\in I_2^{(3)}} \gamma_j d(x, Ax) + \sum_{j\in I_2^{(4)}} \gamma_j d(x_j, Ax)
\end{aligned}
$$

$$+ \sum_{j\in I_2^{(6)}} \gamma_j d(x_j, A^2x_j) + \sum_{j\in I_2^{(7)}} \gamma_j d(x, Ax)$$
$$+ \sum_{j\in I_2^{(8)}} \gamma_j d(Ax_j, A^2x_j) + \sum_{j\in I_2^{(9)}} \gamma_j d(Ax_j, A^2x).$$

After not more than n_0 steps we shall that

$$d(x, Ax) \leq \sum_{j\in I^*} \gamma_j^* d(v_j, Ax) + \gamma_0^* r,$$

for

$$\gamma_j^* \geq 0,\ i \in \{0\} \cup I^*$$
$$\gamma_0^* + \sum_{j\in I^*} \gamma_j^* = 1$$

and

$$v_j \in F_{0,j} \in I^*.$$

Since F_0 is the centre we have that

$$d(v_j, Ax) \leq r,$$

which implies that

$$d(x, Ax) \leq r \text{ for all } x \in F_m.$$

Similarly, we can prove that

$$d(x, Ay) \leq r \text{ for all } x, y \in F_m.$$

So, in the second case we have

$$\begin{aligned} \delta_{m+1} - \varepsilon_n &\leq d(\tilde{x}_n, \tilde{y}_n) \\ &= d(x_n, Ay_n) \leq r, \text{ for } n \in \mathbf{N}, \end{aligned}$$

and consequently

$$\delta_{m+1} \leq r.$$

Using (4) it is easy to prove this inequality for cast (iii). Thus,

$$\delta_m \leq r \text{ for all } m \in \mathbf{N}.$$

Let us define $F^\infty = \bigcup_{k=0}^\infty F_k$.

F_0 is non-empty. So, F^∞ is non-empty too.

Since $\delta(F^\infty) < r\delta(F)$, F^∞ is a closed proper subset of F.

Moreover, W is continuous and that the closure of convex set is convex. Since mapping A is continuous so,

$$A(F^\infty) \subset F^\infty$$

and therefore F^∞ is a subset of F, which is a contradiction to the minimality of F. Hence, F consists of a single element which is a fixed point for A.

References

[1] Guay, M. D., Singh, K. L., Fixed point of set valued mapping of convex metric spaces, Jnanabha 16 (1986), 13-22.

[2] Hadžić, O., Gajić, Lj., Coincidence points for set valued mappings in convex metric spaces, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 16 (1986), 11-25.

[3] Machado, H. V., A characterization of convex subsets of normed spaces, Kodai Math. Sem. Rep. 25 (1973), 307-320.

[4] Naimpally, S. A., Singh, K. L., Fixed point and common fixed points in convex metric space, Preprint.

[5] Takahashi, W., A convexity in metric space and non-expansive mappings 1, Kodai Math. Sem. Rep. 22 (1970), 142-149.

[6] Tallman, L. A., Fixed point for condensing multifunctions in metric space with convex structure, Kodai Math. Sem. Rep. 29 (1979), 62-70.

[7] Vajzović, F., Fixed point theorems for nonlinear operators, Radovi Mat. 1 (1985), 49-59 (in Russian).

[8] Gajić, Lj., On convexity in convex metric spaces with applications, J. Nature Phys. Sci. 3 (1989), 39-48.

Received by the editors October 10, 1992.