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A FIXED POINT THEOREM IN ABSTRACT ECONOMY

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Abstract

We prove a generalization of Himmelberg's fixed point theorem, and as application get the existence of equilibrium point for abstract economies.

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1. Introduction

In the last twenty years the classical results on the existence of equilibrium has been generalised in many directions ([3], [6]) for compact and convex underlying spaces and in ([4], [14]) for paracompact not necessery compact spaces.

The purpose of this paper is first to prove a generalization of D. I. Rim's and W. K. Kim's generalization of Himmelberg's fixed poit theorem by relaxing the condition on local convexity of underlying space, and second, to prove some applications of this result on existence of equilibria.

2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all nonempty subsets of A and by $cl\ A$ the closure of A in X.If A is a subset of a topological vector space, we shall denote by $convA$ the convex hull of A.If A is a nonempty subset of a topological vector space X and S, T: $A \to 2^X$ are the corespondences, then coT, clT,$T \cap S : A \to 2^X$ are correspondences defined by $(coT)(x) = convT(x), (clT)(x) = clT(x)$ and $(S \cap T)(x) = T(x) \cap S(x)$ for each $x \in A$, respectively. Let B be a nonempty subset of A. Denote the restriction of T on B by $T|_B$. Let X be a nonempty subset of topological vector space and $x \in X$. Let $\phi : X \to 2^X$ be a given correspondence. A correspondence $\phi_x :\to 2^X$ is said to be Θ-majorant of ϕ at x if there exists an open neighbourhood N_x in X such that (a) for each $z \in N_x, \phi(z) \subset \phi_x(z)$, (b) for each $z \in N_x, z \notin clco\ \phi_x(z)$ and (c) $\phi_x|_{N_x}$ has open graph in $N_x \times X$. The correspondence ϕ is said to be Θ-majorised if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists a Θ-majorant of ϕ at x. It is clear that every correspondence ϕ having an open graph with $x \notin clco\phi(x)$ for each $x \in X$ is Θ-majorised correspondence.

In an example of $\Theta-$majorised mapping which does not have an open graph is given. Let X and Y be two topological spaces.

Definition 1. *A correspondence (multivalued function)* $T : X \to 2^X$ *is said to be* **almost upper semicontinuous** *if for each* $x \in X$ *and each open set* V *in* Y *with* $T(x) \subset V$, *there exists an open neighbourhood* U *of* x *in* X *such that* $T(y) \subset clV$.

An upper semicontinuous (*u.s.c.*) correspondence is clearly almost upper semicontinuous but converse is not thrue. (For example see [11]). But, with some additional conditions an almost upper semicontinuous function can be upper semicontinuous.

Lemma 1. *([11]) Let* X *be a nonempty subset of topological space* E, *and* D *be a nonempty compact subset of* X. *Let* $T : X \to 2^D$ *be an almost upper semicontinuous function such that for each* $x \in X$, $T(x)$ *is closed. Then* T *is an upper semicontinuous multivalued function.*

For any upper semicontinuous function T: $X \to 2^Y$ coT and $clcoT$ are not necessarily *u.s.c.* but we shall prove that the almost upper semicontinuity can be preserved if $T(X)$ is a so-called $Z-$type subset of not necesserily locally convex Hausdorff topological vector space.

Definition 2. *Let X be a subset of a Hausdorff topological vector space $K \subset X$ and $\mathbf{U}$ the fundamental system of neighbourhoods of zero in X. The set K is said to be of Z-**type** if for every $V \in \mathbf{U}$ there exist $U \in \mathbf{U}$ such that*

$$conv(U \cap (K - K)) \subset V.$$

Lemma 2. *Let $\{X_i\}_{i \in I}$ be a family of nonempty compact convex subsets of Hausdorff topological vector spaces $\{E_i\}_{i \in I}$ ($K_i \subset E_i, i \in I$). $E = \prod_{i \in I} E_i$ and $X = \prod_{i \in I} X_i$. If for every $i \in I$ the set X_i is of Z-type in E_i, then X is of Z-type in E.*

Lemma 3. *Let X be a convex Z-type subset. Then, for every $U \in \mathbf{U}$ there exists $V \in \mathbf{U}$ so that*

$$conv((C + V) \cap X) \subset C + U$$

for every compact subset C of X.

In [7], a nontrivial example of Z-type subset of metric topological vector space is given.

Now we recall the following general definitions of equilibrium theory in mathematical economics. Let I be a finite or infinite sets of agents. For each $i \in I$, let X_i be a nonempty set of actions. **An abstract economy** (or **generalized game**) $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ is defined as a family of ordered quadruples (X_i, A_i, B_i, P_i) where X_i is a nonempty topological vector space (a choise set), A_i,B_i : $\prod_{j \in I} X_j \to 2^{X_i}$ are constraint correspondences and the $P_i : \prod_{j \in I} X_j \to 2^{X_i}$ is a preference correspondence. An **equilibrium** for Γ (Shafer-Sonneinschein type) is a point $\hat{x} \in X = \prod_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in clB_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. When $A_i = B_i$, for each $i \in I$, our definitions of an abstract economy and an equlibrium coincide with the standard definition of Shafer-Sonnenschein or in [12]. For each $i \in I$, $P_i' : X \to 2^X$ will denote the correspondence defined by $P_i' = \{y \in X : y_i \in P_i(x)\}$. We shall use the following notation:

$$X^i = \prod_{j \in I, j \neq i} X_j$$

and let $\pi_i : X \to X_i$,$\pi^i : X \to X^i$, be the projections of X onto X andX^i, respectively. For any $x \in X$, we simply denote $\pi_i(x) \in X^i$ by x^i and $x = (x^i, x_i)$.

J. Greenberg introduced a further generalised concept of equilibrium as follows: Under the same settings as above, let $\Psi = \{\psi_i\}_{i\in I}$ be a family of functions $\psi_i : X \to R^+$ for each $i \in I$. A **Ψ-quasi-equilibrium** for Γ is a point $\hat{x} \in X$ such that for all $i \in I$

1. $\hat{x}_i \in clA_i(\hat{x})$,
2. $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$ and/or $\psi_i(\hat{x}) = 0$

As remarked in [11], quasi-equilibrium can be of special interest for economies with a tax authority and result of Shafer-Sonnenschein can not be applied to this problem.

Now we give another definition of equilibrium for abstract economy by utility functions. By following Debreu [1], an abstract economy $\Gamma = (X_i, A_i, f_i)_{i\in I}$ is defined as a family of ordered triples (X_i, A_i, f_i) where X_i is a nonempty topological vector space (a choice set), $A_i : \prod_{j\in I} X_j = X \to R$ is a utility function (payoff function). An **equilibrium** for Γ (Nash type) is a point $\hat{x} \in X$ such that for each $i \in I$,$\hat{x}_i \in A_i(\hat{x})$ and

$$f_i(\hat{x}) = f_i(\hat{x}^i, \hat{x}_i) = \inf\{f_i(\hat{x}_1, ..., \hat{x}_{i-1}, z, \hat{x}_{i+1}, ...)|z \in clA_i(\hat{x})\}.$$

It should be noted that if $A_i(x) = X_i$ for all $x \in X$, then the concept of an equillibrium for Γ coincides with the well-know Nash equilibrium. And as is remarked in [11], two types of equilibrium points coincide when the preference correspondence P_i can be defined by

$$P_i(x) = \{z_i \in X_i | f_i(x^i, z_i) < f_i(x)\}$$

for each $x \in X$.

3. A generalization of Hadžić's fixed point theorem

We begin with the following result which is a generalization of Lemma 2 from [11].

Lemma 4. *Let X be a convex subset of Hausdorff topological space E and let D be a nonempty compact subset of X. Let $T : X \to 2^D$ be an almost upper semicontinuous correspondence such that for each $x \in X$, $convT(x) \subset D$ and $clT(X)$ is of Z-type subset. Then $clcoT$ is u.s.c..*

Proof. For any $x \in X$ let U be an open set containing $clconvT(x)$. Since $clconvT(x)$ is compact in D we can find an open neighbourhood N of 0 such that

$$clconvT(x) + N \subset clconvT(x) + clN \subset U.$$

By Lemma 2 there exists an open zero neighbourhood $N_1 \subset N$ such that

$$conv((clconvT(x) + clN_1) \cap clT(X)) \subset clconvT(X) + N.$$

Clearly, $V = clconvT(x) + N_1$ is an open set containing $clconvT(x)$ and $V \subset U$. Since T is almost upper semicontinuous there exists an open neighbourhood W of x in X such that $T(y) \subset clV$ for all $y \in W$. Then

$$clconvT(y) = clconv(T(y) \cap clT(X)) \subset$$

$$clconv(cl(cl(convT(x) + N_1) \cap clT(X))$$

$$\subset clconv((clconvT(x) + N_1) \cap clT(X)) \subset cl(clconvT(x) + N)$$

$$\subset clconvT(x) + clN \subset U$$

for all $y \in W$. The proof is complete.

We now prove the following generalization of Hadžić's fixed point theorem.

Theorem 1. *Let X be a convex subset of Hausdorff topological vector space E and D be a nonempty compact subset of X. Let $S, T : X \to 2^D$ be almost upper semicontinuous functions such that*

1. *for each $x \in X, \emptyset \neq convS(x) \subset T(x)$;*
2. *for each $x \in X$, $T(x)$ is closed;*
3. *$clT(X)$ is of Z-type subset.*

Then, there exists a point $\hat{x} \in D$ such that $\hat{x} \in T(\hat{x})$.

Proof. For each $x \in X$ we have that $clconvS(x) \subset T(x)$. By Lemma 1 $clcoS$ is upper semicontinuous and closed convex-valued in D. Therefore, by Hadžić's fixed point theorem [7], there exists a point $\hat{x} \in D$ such that $\hat{x} \in clconvS(\hat{x})$, which completes the proof.

Corollary 1. *Let X be a convex subset of Hausdorff topological vector space E and D be a nonempty compact subset of X. Let $S : X \to 2^D$ be an almost upper semicontinuous function such that for each $x \in X$, $convS(x)$ is a nonempty subset of D and set $clS(X)$ is of Z-type. Then, there exists a point $\hat{x} \in D$ such that $\hat{x} \in clcoS(\hat{x})$.*

4. Existence of equilibria in abstract economies

In this section we consider both kinds of economy described in the preliminaries (that is, an abstract economy given by preference correspondences (Shafer-Sonnenshein type) in compact settings and abstract economy given by utility functions (Nash type) in noncompact settings) and prove the existence of equilibrium points or quasi-equilibrium points for either case by using the fixed point theorems in section 3.

First, using Θ-majorised correspondences we shall prove an equilibrium existence of a compact abstract economy, which generalises result of D.I.Rim and W.K. Kim. For simplicity, we may assume that $A_i = B_i$ for each $i \in I$ in an abstract economy.

Theorem 2. *Let $\Gamma = (X_i, A_i, P_i)$ be an abstract economy where I is a countable set such that for each $i \in I$,*

1. *X_i is a nonempty compact Z-type subset of metrisable Hausdorff topological vector space,*
2. *for each $x \in X = \prod_{i \in I} X_i$, $A_i(x)$ is a nonempty convex,*
3. *the correspondence $clA_i : X \to 2^{X_i}$ is continuous,*
4. *the correspondence P_i' is Θ-majorised.*

Then Γ has an equilibrium choice $\hat{x} \in X$, that is for each $i \in I$ $\hat{x}_i \in clA_i(\hat{x}_i)$ and $A_i(\hat{x}) \cap P_i(x) = \emptyset$.

Proof. Let $i \in I$ be fixed. Since P_i' is Θ-majorised, for each $x \in X$, as in [11], there exists a correspondence $\phi_x : X \to 2^{X_i}$ and an open neighbourhood U_x of x in X such that $P_i(z) \subset \phi_x(z)$ and $z_i \notin clconv\phi_x$ for each $z \in U_x$, and $\phi|_{U_x}$ has an open graph in $U_x \times X_i$. By the compactness of X, the family $\{U_x : x \in X\}$ contains a finite subcover $\{U_{x_j} : j \in J\}$, where $J = \{1, 2, ..., n\}$. For each $j \in J$, we now define $\phi_j : X \to 2^{X_i}$ by

$$\phi_j(z) = \begin{cases} \phi_{x_j}(z), & \text{if } z \in U_{x_j}, \\ X_i, & \text{if } z \notin U_{x_j}. \end{cases}$$

and next we define $\Phi_i : X \to 2^{X_i}$ by

$$\Phi_i(z) = \cap_{j \in J} \phi_j(z), \text{ for each} z \in X.$$

For each $z \in X$, there exists $k \in J$ such that $z \in U_{x_k}$ so that $z_i \notin clconv\phi_{x_k}(z) = clconv\phi_k(z)$; thus $z_i \notin clconv\Phi_i(z)$. As in [11] one can show that there exists a continuous function $C_i : X \times X_i \to [0, 1]$ such that $C_i(x, y) = 0$ for all $(x, y) \notin graph$ of Φ_i and $C_i(x, y) \neq 0$ for all $(x, y) \in graph$ of Φ_i. For each $i \in I$, we define a correspondence $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \{y \in clA_i(x) : C_i(x, y) = max_{z \in clA_i} C_i(x, z)\}.$$

The correspondence F_i is upper semicontinuous [2]. So, a correspondence $G : X \to 2^X$ defined by $G(x) = \prod_{i \in I} F_i(x)$ is also upper semicontinuous and $G(x)$ is a nonempty compact subset of X for each $x \in X$. Since by Lemma 2 X is of Z-type subset one can use Corollary 1. Now as in [11] one can prove that the fixed point of G is the equilibrium choice for Γ.

Using the concept of Ψ-quasi-equilibrium described in the preliminaries, we further generalise Theorem 2 as follows:

Theorem 3. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be an abstract economy where I is a countable set such that for each $i \in I$,*

1. *X_i is a nonempty compact convex Z-type subset of a metrisable Hausdorff topological vector space,*
2. *$\psi_i : X = \prod_{i \in I} X_i \to R^+$ is a non-negative real-valued lower semicontinuous function,*
3. *for each $x \in X$, $A_i(x)$ is nonempty convex,*

4. *the correspondence* $clA_i : X \to 2^{X_i}$ *is continuous for all* x *with* $\psi_i(x) > 0$ *and is almost upper semicontinuous for all* x *with* $\psi_i(x) = 0$,

5. *the correspondence* P_i' *is* Θ – *majorised*

Then, Γ *has a* Ψ – *quasi* – *equilibrium choice* $\hat{x} \in X$, *that is, for each* $i \in I$,

1. $\hat{x}_i \in clA_i(\hat{x})$,

2. $A_i(\hat{x}) \cap P_i(\hat{x}) = 0$ *and/or* $\psi_i(\hat{x}) = 0$

Proof. See [11].

Finally, using Theorem 1 again, we can prove the quasi-equilibrium existence theorem of the Nash type non-compact abstract economy.

Theorem 4. *Let* I *be any (possible uncountable) index set and for each* $i \in I$ *let* X_i *be a convex subset of Hausdorff topological vector space* E_i *and* D_i *be a nonempty compact* Z*-type subset of* X_i. *For each* $i \in I$, *let* $f_i : X = \prod_{i \in I} X_i \to R$ *be a continuous function and* $\psi : X \to R^+$ *be a non-negative real-valued lower semicontinuous function. For each* $i \in I$, $S_i : X \to 2^{D_i}$ *be continuous correspondence for all* $x \in X$ *with* $\psi_i(x) > 0$ *and be almost upper semicontinuous for all* $x \in X$ *with* $\psi_i(x) = 0$ *such that*

1. $S_i(x)$ *is a nonempty closed subset of* D_i,

2. $x_i \to f_i(x^i, x_i)$ *is quasi-convex on* $S_i(x)$.

Then there exists an equilibrium poit $\hat{x} \in D = \prod_{i \in I} D_i$ *such that for each* $i \in I$,

1. $\hat{x}_i \in S_i(\hat{x})$,

2. $f_i(\hat{x^i}, \hat{x}_i) = inf_{z \in S_i(\hat{x})} f_i(\hat{x^i}, z)$ *and/or* $\psi_i(\hat{x}) = 0$.

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