Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 1 (1995), 29-38

Review of Research
Faculty of Science
Mathematics Series

g-FUNCTIONS

Ján Rybárik
Technical University, Department of Mathematics,
031 19 Liptovský Mikuláš, Slovakia

Abstract

In this paper some elementary g-functions are derived as solutions of some functional equations using several results of Aczel [1]. Applying pseudo-arithmetical operations to these functions we obtain a wider class of g-function. Further we will apply the g-derivative [5] to these g-function and deduce relations between some properties of g-functions and their g-derivatives.

AMS Mathematics Subject Classification (1991): 28A15
Key words and phrases: g - derivative, g - function.

1. Introduction

Pseudo-arithmetical operations introduced in [2] and [4] and investigated in [3] are useful tools in treating nonlinear problems (see [5], [6]). Among the basic concepts, which are necessary to be built, are the modified functions - they will be called the g-functions. Some elementary g-functions are derived as solutions of some functional equations using several results of Aczel [1]. Applying pseudo-arithmetical operations to these functions we obtain a wider class of g-function. This calculus is a further development of g-calculus for the real functions introduced in [5] and investigated in [7]. Further we will apply the g-derivative [5] to these g-function and we deduce relations between some properties of g-functions and their g-derivatives.

2. g-calculus for the functions

A consistent system of pseudo-arithmetical operations may be used to the creation of functions so that we replace common arithmetical operations by pseudo-arithmetical operations in ruling. We can obtain directly the rational functions only and that is why we introduce some elementary functions as the solutions of corresponding functional equations.

We will work with the real function f, which is continuous on (a,b) and $(a,b) \subset (-\infty,+\infty)$. Further, let a function $g : (-\infty,+\infty) \to (-\infty,+\infty)$ be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$ (see [3]). Then g is a continuous monotone strictly increasing unbounded odd function, $g(0) = 0$, $g(1) = 1$, and

$$\begin{aligned} x \oplus y &= g^{-1}(g(x) + g(y)), \\ x \ominus y &= g^{-1}(g(x) - g(y)), \\ x \odot y &= g^{-1}(g(x) \cdot g(y)) \text{ and} \\ x \oslash y &= g^{-1}(g(x) / g(y)),\ y \neq 0. \end{aligned}$$

The function corresponding to a function f introduced by the g-calculus will be called the **g-function** and we denote it by f_g (see Definition 3).

Definition 1. *A continuous function f_g such that it is a solution of the functional equation*

$$f_g(x) \oplus f_g(y) = f_g(x \odot y) \quad \textit{and} \quad f_g(g^{-1}(a)) = 1, \quad \textit{where} \quad a > 0,\ a \neq 1$$

will be called **the g-logarithmic function** *and denoted by* $\underset{\mathrm{g}}{\log_a} x$.

Theorem 1. *For every $x \in (0,+\infty)$ it holds*

$$\underset{\mathrm{g}}{\log_a} x = g^{-1}(\log_a g(x)). \tag{1}$$

Proof. The requirement that $x \in (0,+\infty)$ follows from the properties of a logarithmic function.

Using Theorem 2 and Corollary 3.4 [3] we can rewrite the equation $f_g(x) \oplus f_g(y) = f_g(x \odot y)$ in the form

$$g^{-1}(g(f_g(x)) + g(f_g(y))) = f_g(g^{-1}(g(x) \cdot g(y))).$$

Because of $g^{-1}(g(z)) = z$ we have

$$g^{-1}(g(f_g(x)) + g(f_g(y))) = g^{-1}(g(f_g(g^{-1}(g(x) \cdot g(y)))))$$

and further

$$g(f_g(x)) + g(f_g(y)) = g(f_g(g^{-1}(g(x) \cdot g(y)))).$$

Now, putting $x = g^{-1}(g(x))$ and $y = g^{-1}(g(y))$ we obtain

$$g(f_g(g^{-1}(g(x)))) + g(f_g(g^{-1}(g(y)))) = g(f_g(g^{-1}(g(x) \cdot g(y)))).$$

If we denote $g(x) = u$, $g(y) = v$ and $g(f_g(g^{-1}(z))) = \Phi(z)$, then we can express the last functional equation as

$$\Phi(u) + \Phi(v) = \Phi(u \cdot v),\ \Phi(a) = 1.$$

According to Aczel [1], the logarithmic function is the only continuous solution of this functional equation, i.e. $\Phi(x) = \log_a x$, $a > 0$, $a \neq 1$. Thus $g(f_g(g^{-1}(x))) = \log_a x$, $a > 0$ and from this follows that

$$f_g(x) = g^{-1}(\log_a g(x)). \text{ Hence } \underset{\text{g}}{\log_a x} = g^{-1}(\log_a g(x)). \ \square$$

Definition 2. *A continuous function f_g such that it is a solution of the functional equation*

$f_g(x) \odot f_g(y) = f_g(x \oplus y)$, $f_g(1) = g^{-1}(a)$, where $a > 0$, $a \neq 1$ will be

called **a g-exponential function** *and denoted by* $\underset{\text{g}}{a^x}$.

Theorem 2. *For every $x \in (-\infty, +\infty)$ it holds*

$$\underset{\text{g}}{a^x} = g^{-1}(a^{g(x)}). \tag{2}$$

Proof. We use the same technique as applied to prove Theorem 1, and we employ the fact that the functions $\phi :\ \phi(x) = a^x,\ a > 0$ are the only continuous solution of the functional equation $\Phi(u)\cdot\Phi(v) = \Phi(u+v),\ \Phi(1) = a.$□

Theorem 3. *The g-exponential function is an inverse function of a g-logarithmic function.*

Proof. The g-logarithmic function is given by the formula $y = g^{-1}(\log_a g(x))$. From whence it follows that $g(y) = \log_a g(x)$ then $a^{g(y)} = g(x)$ and $x = g^{-1}(a^{g(y)})$. Now, if we apply (2) , we have $x = \underset{g}{a^y}$.□

Remark 1. Using the same aproach we can introduce **the g-power function** which will be denoted by $y = \underset{g}{x^r}$, $r > 0$. It is one of the increasing continuous solutions of the functional equation

$$f_g(x) \odot f_g(y) = f_g(x \odot y).$$

This function is given by

$$\underset{g}{x^r} = g^{-1}((g(x))^r),\ r > 0, \quad \text{where } x \in [0, +\infty). \tag{3}$$

Now, we shall genaralize the results achieved above.

Definition 3. *Let f be a continuous function on (a, b), where $(a, b) \subseteq (-\infty, \infty)$ and the function g be a generator of the consistent system of pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$. The function f_g given by*

$$f_g(x) = g^{-1}(f(g(x))) \quad \textit{for every} \quad x \in (g^{-1}(a), g^{-1}(b)) \tag{4}$$

is said to be **the g-function** *corresponding to the function f. This fact is denoted by*

$$f \vdash^g f_g .$$

Remark 2. It is easy to see that, for any f, it is

$$(f_g)^{-1} = (f^{-1})_g ,$$

(if at last one side of this equality exists).

Example 1. Let g be a function given by $g(x) = x^3$, where $x \in (-\infty, +\infty)$ and the function f is defined on the interval $(-\infty, +\infty)$ by $f(x) = (1 + x)^3$. The consistent system of the pseudo-arithmetical operations which is generated by this function g will be the system $\{\oplus, \ominus, \odot, \oslash\}$ such that

$$\begin{aligned} x \oplus y &= \sqrt[3]{x^3 + y^3} & x \odot y &= x \cdot y \\ x \ominus y &= \sqrt[3]{x^3 - y^3} & x \oslash y &= \tfrac{x}{y} . \end{aligned}$$

Then the g-function f_g corresponding to the function f is defined by $f_g(x) = \sqrt[3]{(1 + x^3)^3} = 1 + x^3$, where $x \in (-\infty, +\infty)$.

Theorem 4. *Let the function g be a generator of the consistent system of the pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$. Let f and h be continuous functions on (a, b), where $(a, b) \subset (-\infty, +\infty)$ and $\alpha \in (-\infty, +\infty)$ is a constant. Then we have*

$$\begin{array}{lrcl} (a) & \alpha \cdot f(x) & \vdash^g & g^{-1}(\alpha) \odot f_g(x) , \\ (b) & f(x) + h(x) & \vdash^g & f_g(x) \oplus h_g(x) , \\ (c) & f(x) - h(x) & \vdash^g & f_g(x) \ominus h_g(x) , \\ (d) & f(x) \cdot h(x) & \vdash^g & f_g(x) \odot h_g(x) , \\ (e) & \frac{f(x)}{h(x)} & \vdash^g & f_g(x) \oslash h_g(x) , \end{array}$$

for every $x \in (g^{-1}(a), g^{-1}(b))$.

Proof. We shall prove only the formula (b) because the proof of other formulas is based on the same principles. Put $k(x) = f(x) + h(x)$. Using (4) and the definition of the pseudo-addition [3] we have

$$\begin{aligned} k_g(x) &= g^{-1}[k(g(x))] = g^{-1}[\, f(g(x)) + h(g(x)) \,] = \\ &= g^{-1}[\, g(g^{-1}(f(g(x)))) \;+\; g(g^{-1}(h(g(x)))) \,] = \\ &= g^{-1}[g(f_g(x)) + g(h_g(x))] = \; f_g(x) \oplus h_g(x) . \end{aligned}$$

Hence $k(x) = f(x) + h(x) \vdash^g k_g(x) = f_g(x) \oplus h_g(x)$□.

Corollary 1. *The g-function f_g corresponding to the function f can be expressed by using the pseudo-arithmetical operations, so that we replace in*

the ruling of the function f :
(1) every constant α by the constant $g^{-1}(\alpha)$,
(2) every elementary function by the g-elementary function,
(3) every arithmetical operation by the corresponding pseudo-arithmetical operation.

Example 2. Let it be required to find the g-function corresponding to the function $f : f(x) = x \cdot \log \frac{1}{x}$ where $x \in (0, +\infty)$. Using (4) we have

$$f_g(x) = g^{-1}(g(x) \cdot \log \frac{1}{g(x)}) \text{ for every } x \in (0, +\infty).$$

Now, we show that this formula can be expressed in the way described in Corollary 1.
Then $f_g(x) = g^{-1}(g(x) \cdot \log \frac{1}{g(x)}) = g^{-1}(g(x) \cdot \log \frac{1}{g(x)}) =$

$$= g^{-1}(g(x) \cdot g(g^{-1}(\log g(g^{-1}(\tfrac{1}{g(x)}))))) = g^{-1}(g(x) \cdot g(\underset{\mathrm{g}}{\log} (g^{-1}(\tfrac{g(g^{-1}(1))}{g(x)})))) =$$

$$= g^{-1}(g(x) \cdot g(\underset{\mathrm{g}}{\log} (g^{-1}(1) \oslash x))) = x \odot \underset{\mathrm{g}}{\log} (g^{-1}(1) \oslash x).$$ Whence

$$x \cdot \log \frac{1}{x} \vdash^{g} x \odot \underset{\mathrm{g}}{\log} (g^{-1}(1) \oslash x).$$

3. g-derivatives and some properties of g-functions

The notion of a g-derivative built on the operations of the pseudo-addition and pseudo-multiplication was introduced by E. Pap [5]. He applied this notion to solve some nonlinear differential equations. Now we shall give a definition of the g-derivative of the g-functions.

Definition 4. *Let g be a function which generates a consistent system of the pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$ and let g be differentiable on $(-\infty, +\infty)$. Further, let the function f be continuous and differentiable on (a, b), where $(a, b) \subset (-\infty, +\infty)$. Then, the* **g-derivative** *of a function* *f is defined by*

$$\frac{d^{\oplus}}{dx}f(x) = g^{-1}(\frac{d}{dx}g(f(x))) \quad \textit{for every } x \in (a,b).$$

Proposition 1. *The basic properties of the g-derivatives (see [5]) hold for the g-functions too. Then*

$$\begin{array}{lrcl}
(a) & \frac{d^{\oplus}}{dx}(f_g(x) \oplus h_g(x)) & = & \frac{d^{\oplus}}{dx}f_g(x) \oplus \frac{d^{\oplus}}{dx}h_g(x)\,, \\
(b) & \frac{d^{\oplus}}{dx}(\lambda \odot f_g(x)) & = & \lambda \odot \frac{d^{\oplus}}{dx}f_g(x)\,, \lambda \textit{ is constant}\,, \\
(c) & \frac{d^{\oplus}}{dx}(f_g(x) \odot h_g(x)) & = & (\frac{d^{\oplus}}{dx}f_g(x) \odot h_g(x)) \oplus (f_g(x) \odot \frac{d^{\oplus}}{dx}h_g(x))\,, \\
(d) & \frac{d^{n\oplus}}{dx}f_g(x) & = & g^{-1}(\frac{d^n}{dx^n}g(f_g(x)))\,, n \in N\,,
\end{array}$$

where $\frac{d^{n\oplus}}{dx^n}k(x) = \frac{d^{\oplus}}{dx}[\frac{d^{(n-1)\oplus}}{dx^{n-1}}k(x)]$ *is a g-derivative of the n-th order.*

Theorem 5. *If the functions f and g satisfy the assumptions of Definition 4 and they are n-times differentiable on (a,b), $n \in N$, then we have, for every $x \in (g^{-1}(a), g^{-1}(b))$ and $k = 1,2,\ldots,n$,*

$$\frac{d^{k\oplus}}{dx^k}f_g(x) = g^{-1}[\frac{d^k}{dx^k}f(g(x))]. \tag{5}$$

Proof. Using Proposition 1 and (4) we obtain

$$\begin{aligned}
\frac{d^{k\oplus}}{dx^k}f_g(x) &= g^{-1}(\frac{d^k}{dx^k}g(f_g(x))) = g^{-1}(\frac{d^k}{dx^k}g(g^{-1}(f(g(x))))) = \\
&= g^{-1}(\frac{d^k}{dx^k}f(g(x))).\square
\end{aligned}$$

Now, we will show some properties of the g-functions. Consider a function g which is the generator of a consistent system of the pseudo-arithmetical operations $\{\oplus, \ominus, \odot, \oslash\}$ on $(-\infty, +\infty)$ and a function f such that it is continuous on (a,b), where $(a,b) \subset (-\infty,+\infty)$. Let f_g be the g-function corresponding to the function f, i.e.

$$f_g(x) = g^{-1}(f(g(x))) \text{ for every } x \in (g^{-1}(a), g^{-1}(b)).$$

Theorem 6. *The function f_g increases [decreases] on the interval $(g^{-1}(a), g^{-1}(b))$ iff f is an increasing [decreasing] function on the interval (a,b).*

Proof. ($\Rightarrow$) Let f_g be an increasing function on $(g^{-1}(a), g^{-1}(b))$ and $y_1, y_2 \in (a,b)$. Then, $x_1 = g^{-1}(y_1)$ and $x_2 = g^{-1}(y_2)$ lie in the interval $(g^{-1}(a), g^{-1}(b))$. Using the fact that g and g^{-1} are increasing functions we obtain

$$y_1 < y_2 \Rightarrow x_1 = g^{-1}(y_1) < x_2 = g^{-1}(y_2) \Rightarrow f_g(x_1) < f_g(x_2).$$

According to (4) it is
$f_g(x_1) = g^{-1}(f(g(x_1))) = g^{-1}(f(g(g^{-1}(y_1)))) = g^{-1}(f(y_1))$ and similarly $f_g(x_2) = g^{-1}(f(y_2))$. Then

$$y_1 < y_2 \Rightarrow g^{-1}(f(y_1)) < g^{-1}(f(y_2)) \Rightarrow f(y_1) < f(y_2).$$

It means that the function f increases on the interval (a,b).

($\Leftarrow$) If f is an increasing function on (a,b) and $x_1, x_2 \in (g^{-1}(a), g^{-1}(b))$ then $y_1 = g(x_1)$ and $y_2 = g(x_2)$ belong to (a,b). Thus we have
$x_1 < x_2 \Rightarrow y_1 = g(x_1) < y_2 = g(x_2) \Rightarrow f(g(x_1)) = f(y_1) < f(y_2) = f(g(x_2))$. Hence $g^{-1}(f(g(x_1))) < g^{-1}(f(g(x_2)))$, where $g^{-1}(f(g(x_1))) = f_g(x_1)$ and $g^{-1}(f(g(x_2))) = f_g(x_2)$. It means that the following holds $x_1 < x_2 \Rightarrow f_g(x_1) < f_g(x_2)$ for every $x_1, x_2 \in (g^{-1}(a), g^{-1}(b))$ and the function f_g increases on the interval $(g^{-1}(a), g^{-1}(b))$.□

Theorem 7. *Let the function f_g be increasing [decreasing] on the interval $(g^{-1}(a), g^{-1}(b))$. Then we have $\frac{d^{\oplus}}{dx} f_g(x) \geq 0$ $[\frac{d^{\oplus}}{dx} f_g(x) \leq 0]$ for every $x \in (g^{-1}(a), g^{-1}(b))$.*

Proof. Assume f_g is an increasing function on $(g^{-1}(a), g^{-1}(b))$. According to Theorem 6 the function f is increasing on (a,b) and $f'(x) \geq 0$ for every $x \in (a,b)$. Using Theorem 5 we obtain

$$\frac{d^{\oplus}}{dx} f_g(x) = g^{-1}(\frac{d}{dx} f(g(x))) = g^{-1}(f'(g(x)) \cdot g'(x))$$

for every $x \in (g^{-1}(a), g^{-1}(b))$. Further, let $x_0 \in (g^{-1}(a), g^{-1}(b))$ be an arbitrary point. Then $y_0 = g(x_0)$ belongs to (a,b) and

$$\frac{d^{\oplus}}{dx} f_g(x_0) = g^{-1}(f'(g(x_0)) \cdot g'(x_0)) = g^{-1}(f'(y_0) \cdot g'(x_0)).$$

Because of $g'(x_0) \geq 0$ for every $x_0 \in (-\infty, +\infty)$ we have $f'(y_0) \cdot g'(x_0) \geq 0$ and $g^{-1}(f'(y_0) \cdot g'(x_0)) \geq 0$. It means that $\frac{d^{\oplus}}{dx} f_g(x_0) = g^{-1}(f'(y_0) \cdot g'(x_0)) \geq 0$ for every $x_0 \in (g^{-1}(a), g^{-1}(b))$.□

Definition 5. *Let $I \subset (-\infty, +\infty)$ be an interval and the function f_g satisfies*

$$f_g((\lambda \odot x_1) \oplus ((1 \ominus \lambda) \odot x_2)) \le (\lambda \odot f_g(x_1)) \oplus ((1 \ominus \lambda) \odot f_g(x_2))$$

$$[f_g((\lambda \odot x_1) \oplus ((1 \ominus \lambda) \odot x_2)) \ge (\lambda \odot f_g(x_1)) \oplus ((1 \ominus \lambda) \odot f_g(x_2))]$$

for every $x_1, x_2 \in I, x_1 \neq x_2$ and for every $\lambda \in (0,1)$.
Then, the function f_g is said to be **pseudo-convex [pseudo-concave]** *on the interval I.*

Theorem 8. *If a function f is convex [concave] on the interval (c,d), where $(c,d) \subset (a,b)$, then the function f_g is pseudo-convex [pseudo-concave] on the interval $(g^{-1}(c), g^{-1}(d))$.*

Proof. Let a function f be convex on the interval (c,d), i.e.

$$f(\lambda \cdot y_1 + (1-\lambda) \cdot y_2) \le \lambda \cdot f(y_1) + (1-\lambda) \cdot f(y_2) \tag{6}$$

for every $y_1, y_2 \in (c,d)$, $y_1 \neq y_2$ and $\lambda \in (0,1)$. Take arbitrary $x_1, x_2 \in (g^{-1}(c), g^{-1}(d)), x_1 \neq x_2$. Then $y_1 = g(x_1)$ and $y_2 = g(x_2)$ belong to (c,d). Using the definitions of the pseudo-arithmetical operations and (4) we obtain

$$\begin{aligned}
&f_g((\lambda \odot x_1) \oplus ((1 \ominus \lambda) \odot x_2)) = \\
&= f_g[g^{-1}(g(\lambda) \cdot g(x_1)) \oplus (g^{-1}(g(1) - g(\lambda)) \odot x_2)] = \\
&= f_g[g^{-1}(g(\lambda) \cdot g(x_1)) \oplus (g^{-1}((g(1) - g(\lambda)) \cdot g(x_2))] = \\
&= f_g[g^{-1}(g(\lambda) \cdot g(x_1) + (g(1) - g(\lambda)) \cdot g(x_2))] = \\
&= g^{-1}(f(g(\lambda) \cdot g(x_1) + (g(1) - g(\lambda)) \cdot g(x_2))).
\end{aligned}$$

Further we denote by $\alpha = g(\lambda)$. From the properties of the function g if follows that $\alpha \in (0,1)$ and $g(1) = 1$. Then

$$f_g((\lambda \odot x_1) \oplus ((1 \ominus \lambda) \odot x_2)) = g^{-1}(f(\alpha \cdot y_1 + (1-\alpha) \cdot y_2)).$$

According to (6) it is $f(\alpha \cdot y_1 + (1-\alpha) \cdot y_2) \le \alpha \cdot f(y_1) + (1-\alpha) \cdot f(y_2)$ and $g^{-1}(f(\alpha \cdot y_1 + (1-\alpha) \cdot y_2)) \le g^{-1}(\alpha \cdot f(y_1) + (1-\alpha) \cdot f(y_2))$. Now, we make up the right side of this inequality:

$$\begin{aligned}
&g^{-1}(\alpha \cdot f(y_1) + (1-\alpha) \cdot f(y_2)) = \\
&= g^{-1}[g(\lambda) \cdot f(g(x_1)) + (g(1) - g(\lambda)) \cdot f(g(x_2))] = \\
&= g^{-1}[g(\lambda) \cdot g(g^{-1}(f(g(x_1)))) + g(g^{-1}((g(1) - g(\lambda))) \cdot g(g^{-1}(f(g(x_2))))] = \\
&= g^{-1}[g(\lambda) \cdot g(f_g(x_1)) + g(1 \ominus \lambda) \cdot g(f_g(x_2))] = \\
&= g^{-1}[g(\lambda \odot f_g(x_1)) + g((1 \ominus \lambda) \odot f_g(x_2))] = \\
&= (\lambda \odot f_g(x_1)) \oplus ((1 \ominus \lambda) \odot f_g(x_2))
\end{aligned}$$

Thus it will be shown that this inequality has the form $f_g((\lambda \odot x_1) \oplus ((1 \ominus \lambda) \odot x_2)) \leq (\lambda \odot f_g(x_1)) \oplus ((1 \ominus \lambda) \odot f_g(x_2))$ and the function f_g is pseudo-convex on the interval $(g^{-1}(c), g^{-1}(d))$.□

Remark 3. 1. Combining Example 2 and Theorem 8, we see that $f_g(x) = g^{-1}(g(x) \cdot \log \frac{1}{g(x)})$ defines a pseudo-concave g-function. This function can be applied in the study of the entropy of $\oplus$-decomposable probability measures.
2. Note that f_g may be pseudo-convex [pseudo-concave] even it its second g-derivative is not non-negative [non-positive] on $(g^{-1}(c), g^{-1}(d))$. This is caused by the generator g, which is required to be increasing only.

References

[1] Aczel, J., Lectures on Functional Equations and Their Applications, Academic Press, New York, 1969.

[2] Ichinashi, H., Tanaka, H., Asai, K., Fuzzy integrals based on pseudo-additions and multiplications, J. Math. Anal. Appl. 130 (1988), 354-364.

[3] Mesiar, R., Rybárik, J., Pseudo-arithmetical operations, Tatra Mountains, Math. Publ. 2 (1993).

[4] Murofushi, T., Sugeno, M., Pseudo-additive measures and integrals, J. Math. Anal. Appl. 122 (1987), 197-222.

[5] Pap, E., g-calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. (to appear).

[6] Pap, E., Decomposable measures and applications on nonlinear partial differ. equations, Rend. Mat. di Palermo Ser. II-num. 28 (1992).

[7] Ralevič, N., Some new properties of g-calculus, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 24,1 (1994), 139-157.

Received by the editors January 29, 1993.