Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 1 (1995), 39-46

Review of Research
Faculty of Science
Mathematics Series

ADJOINT THEOREM ON SEMI-INNER PRODUCT SPACES OF TYPE (P)

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Abstract

In this paper a version of Adjoint theorem for maps on semi-inner product spaces of type (p) is obtained (originally introduced by B. Nath under the name: generalized semi-inner product.

Some properties of the generalized adjoint, introduced on semi-inner product spaces by D.O. Koehler as related notion to the work of Stampfli on adjoint abelian and iso abelian operators are also explored.

AMS Mathematics Subject Classification (1991): 46C20
Key words and phrases: semi-inner product type (p), K-space, adjoint operator

1. Introduction

Recently, E.Pap [9] established a theorem concerning the boundedness of the adjoint operator for an arbitrary map on inner product spaces. This result (Adjoint theorem) was extended for linear operators to normed vector spaces [2], [13] and to locally convex spaces [12]. Adjoint theorem was employed in the proofs of Closed Graph Theorems [9], [11], [12], [13].

In this paper we shall obtain a version of Adjoint theorem for maps on semi-inner product spaces of type (p) (originally introduced by B. Nath [7] under the name: generalized semi-inner product). These spaces are generalization of Lumer's semi-inner product spaces [6] which includes Krein spaces and Pontrjagin spaces [1], [3]. We shall also explore some properties of the generalized adjoint, introduced on semi-inner product spaces by D.O. Koehler [5] as related notion to the work of Stampfli [14] on adjoint abel iso abelian operators.

2. K-spaces

If X is a topological vector space, a sequence $\{x_k\}$ in X is said to be K-sequence if every subsequence of $\{x_k\}$ has a further subsequence $\{x_k\}$ such that the subseries $\sum_k x_{n_k}$ is convergent to an element of X. A topological vector space X is said to be K-space if every sequence which converges to 0 is K-sequence.

We shall need in the proof of Theorem 2 the following ([2], theorem 2.2)

Basic Matrix Theorem. Let $x_{ij} \in X$ for $i, j \in N$. Suppose $[x_{ij}]$ is a K-matrix, i.e.

(I) $\lim_{i\to\infty} x_{i_j} = x_j$ exists for each j and

(II) for each subsequence $\{m_j\}$ there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that

$$\{\sum_{j=1}^{\infty} x_{in_j}\}$$

is Cauchy. Then $\lim_{i\to\infty} x_{ij} = x_j$ uniformly with respect to j. In particular $\lim_{i\to\infty} x_{ii} = 0$.

3. Adjoint theorem

Let X be a vector space over the field F of real or complex numbers.

If a functional $[x, y], [\cdot,\cdot]$; $X \times X \to F$ satisfies the following conditions

$$(1)\quad [x + y, z] = [x, z] + [y, z],\quad x, y, z \in X,$$

$$(2)\quad [\lambda x, y] = \lambda[x, y],\quad \lambda \in F \text{ and } x, y \in X,$$

$$(3)\quad [x, x] > 0 \text{ for } x \neq 0,$$

$$(4) \quad |\,[x,y]\,| \leq [x,x]^{\frac{1}{p}}[y,y]^{\frac{p-1}{p}}, 1 < p < \infty,$$

then we say $[x, y]$ is a semi-inner product of type (p). A vector space X, together with a semi-inner product of type (p) defined on it, will be called a semi-inner product space of type (p) (s.i.p.s.(p)). By B. Nath [7] a s.i.p.s. (p) becomes a normed space with the norm $[x,x]^{\frac{1}{p}}, 1 < p < +\infty$, and every normed vector space can be made into a s.i.p.s(p) (see also [10]).

A s.i.p.s.(p) X is said to be continuous if for every $x, y \in X$ such that $||x|| = ||y|| = 1$

$$Re[y, x + \lambda y] \rightarrow Re[y, x]$$

for all real $\lambda \rightarrow 0$.

A s.i.p.s.(p) X is with the homogeneity property if

$$[x, \lambda y] = |\lambda|^{p-2}\bar{\lambda}[x, y]$$

for all $x, y \in X$ and all $\lambda \in F$. By [10] we have the Riesz Representation: in a continuous s.i.p.s.(p) X with the homogeneity property and which is uniformly convex and complete in the corresponding norm, for every functional $f \in X^*$ there exists a unique element $y \in X$ such that

$$f(x) = [x, y] \quad (x \in X).$$

This induces a duality map A from X into X^* in the sense of Stampfli [14].

Theorem 1. *Let A be a map from s.i.p.s.(p) X with the homogeneity property into X^* given by*

$$A(y) = [\cdot, y].$$

A is one-to-one and onto with the properties

$$||A(y)|| = ||y||^{p-1} \quad and \quad ||A^{-1}y^*|| = ||y^*||^{\frac{1}{p-1}}.$$

Proof. Riesz Representation theorem implies that A is one-to-one and onto. Let $A(y) = y^*$.

$$||A(y)|| = ||y^*|| = \sup_{||x|| \leq 1} |\,[x,y]\,| \leq \sup_{||x|| \leq 1} ||x|| \cdot ||y||^{p-1} = ||y||^{p-1}.$$

On the other side we have

$$||y^*|| \geq y^*(\frac{y}{||y||}) = [\frac{y}{||y||}, y] = ||y||^{p-1}.$$

Hence

$$\|A(y)\| = \|y^*\| = \|y\|^{p-1} \text{ for all } y \in X.$$

This implies $\|y^*\| = \|A^{-1}(y^*)\|^{p-1}$ for $y = A^{-1}(y^*)$.

The adjoint operator T^* for a map $T : X \to Y$ is defined by the following: the domain $D(T^*)$ of T^* is

$$D(T^*) = \{y^* \in Y^* : y^*T \text{ is continuous on } X\} \text{ and}$$

$T^* : D(T^*) \to X^C$ is defined by $T^*y^* = y^*T$, where X^C is the space of all continuous functionals X.

Theorem 2. *Let X be a s.i.p.s.(p) and Y a continuous s.i.p.s.(q) complete and uniformly convex, with the homogeneity property. Let $T : X \to Y$ be an arbitrary map. If X is a K-space, then for the adjoint T^* of T there exists $M > 0$ such that*

$$\sup_{\|x\|\le 1} \|y^*\|^{-1} \cdot | T^*(y^*)(x) | < M$$

for all $y^* \in D(T^*)$, $y^* \neq 0$. If $D(T^*) = Y^*$, then as a consequence it holds that $| [T(x), y] | \le M\|y\|^{q-1}$ for all $y \in Y$ and $\|x\| \le 1$.

Proof. Let $\{y_n^*\}$ be an arbitrary sequence from $D(T^*)$, $\{x_n\}$ an arbitrary sequence in X such that $\|x_n\| \le 1$ and $\{\alpha_n\}$ an arbitrary sequence of numbers such that $\alpha_n \to 0$. Let $t_n = \sqrt{|\alpha_n|}$ and $u_n = \frac{\alpha}{\sqrt{|\alpha_n|}}$ if $\alpha_n \neq 0$ and $u_n = 0$ for $\alpha_n = 0$.

We define the matrix $[x_{ij}]$ in the following way

$$x_{ij} = \begin{cases} t_i \frac{T^*(y_i^*)(u_j x_j)}{\|y_i^*\|}, & y_i^* \neq 0, \\ 0, & y_i^* = 0. \end{cases}$$

We shall prove that $[x_{ij}]$ is a K-matrix.

We have by Riesz representation for $y_i^* \neq 0$

$$|x_{ij}| = t_i \frac{|T^*(y_i^*)(u_j x_j)|}{\|y_i^*\|} = t_i \frac{| < y_i^*, T(u_j x_j) > |}{\|y_i^*\|} =$$

$$= t_i \frac{| [T(u_j x_j), y_i] |}{\|y_i\|^{q-1}} \le t_i \frac{\|T(u_j x_j)\| \, \|y_i\|^{q-1}}{\|y_i\|^{q-1}} =$$

$$= t_i \|T(u_j x_j)\|.$$

Hence $\lim_{i\to\infty} x_{ii} = 0$ for all $i \in N$.

Since $\lim_{j\to\infty} u_j x_j = 0$ and X is a K-space for each subsequence $\{m_j\}$ of natural numbers there exists a subsequence $\{n_j\}$ such that

$$\lim_{k\to\infty} \sum_{j=1}^{k} u_{n_j} x_{n_j} = x$$

for some $x \in X$.

Then

$$\lim_{k\to\infty} \sum_{j=1}^{k} x_{inj} = \lim_{k\to\infty} t_i \|y_i^*\|^{-1} T^*(y_i^*)(\sum_{j=1}^{k} u_{n_j} x_{n_j}) =$$

$$= t_i \cdot \|y_i^*\|^{-1}\, T^*(y_i^*)(x).$$

We have

$$|\sum_{j=1}^{\infty} x_{inj}| = t_i \|y_i^*\|^{-1}\, |T^*(y_i^*)(x)| =$$

$$= t_i \cdot \|y_i^*\|^{-1} | < y_i^*, T(x) > | =$$

$$= t_i \cdot \|y_i\|^{-q+1} |\, [T(x), y_i]| \leq$$

$$\leq t_i \|y_i\|^{-q+1} \|T(x)\| \cdot \|y_i\|^{q-1} = t_i \cdot \|T(x)\|.$$

Letting $i \to \infty$ we obtain

$$\lim_{i\to\infty} |\sum_{j=1}^{\infty} x_{inj}| = 0,$$

i.e. $\{\sum_{j=1}^{\infty} x_{inj}\}$ is a Cauchy sequence. By the Basic Matrix Theorem $\lim_{n\to\infty} x_{nn} = 0$, i.e.

$$\lim_{n\to\infty} \alpha_n \cdot \|y_n^*\|^{-1} T^*(y_n^*)(x_n) = 0.$$

Hence there exists $M > 0$ such that

$$\sup_{\|x\|\leq 1} \|y^*\|^{-1} \cdot |T^*(y^*)(x)| < M$$

for all $y^* \in D(T^*), y^* \neq 0$.

4. The generalized adjoint

Let X be an s.i.p.s.(p) and Y an s.i.p.s.(q). Let T be an arbitrary map from X into Y. We define its generalized adjoint map T^+ in the following way: the domain $D(T^+)$ of T^+ consists of those $y \in Y$ for which there exists $z \in X$ such that

$$[T(x), y]_Y = [x, z]_X$$

for each $x \in X$ and $z = T^+(y)$. T^+ is a map from $D(T^+)$ into X with the non-empty domain $D(T^+)$, since $0 \in D(T^+)$. Hence $T^+(0) = 0$. If X and Y are Hilbert spaces then the generalized adjoint is the usual adjoint operator. In general, T^+ is not linear even for T bounded linear operator. But it still has some analogous properties of the usual adjoint operator.

Theorem 3. *Let X be an s.i.p.s.(p) and Y a continuous s.i.p.s.(q), complete and uniformly convex both with the homogeneity property and $T : X \to Y$ bounded linear operator.*

Then we have:

(a) $D(T^+) = Y$.

(b) $(\lambda T)^+ = |\lambda|^{q-2}\bar{\lambda} T^+$ *for* $\frac{1}{p} + \frac{1}{q} = 1$.

(c) $T^+ = A^{-1}T^*B$, *where A and B are the duality maps on X and Y, respectively .*

(d) $(TU)^+ = U^+T^+$
$U : Y \to Z$ bounded linear operator and Z continuous s.i.p.s.(r), complete and uniformly convex with homogenous property.

Proof.

(a) Let y be an arbitrary but fixed element from Y. The functional

$$f_y(x) = [T(x), y]_Y$$

is continuous and linear. Hence by Riesz Representation theorem there exists $z \in X$ such that $f_y(x) = [x, z]$, i.e. $y \in D(T^+)$.

(b) We have

$$[x, (\lambda T)^+(y)] = [(\lambda T)(x), y] = \lambda[T(x), y] =$$

$$= |\lambda|^{(q-2)(p-2)} \cdot |\lambda|^{(p-2)} \cdot |\lambda|^{(q-2)} \cdot \lambda[x, T^+(y)] =$$
$$= [x, |\lambda|^{(q-2)}\bar{\lambda}T^+(y)]$$

for all $x \in X$ and $y \in D(T^+)$. Hence $(\lambda T)^+(y) = |\lambda|^{q-2}\bar{\lambda}T^+(y)$ for $y \in D(T^+)$.

(c)

$$< T^*B(y), x >=< T^*(y^*), x >=< y^*, T(x) > =$$
$$[T(x), y] = [x, T^+(y)] =< A(T^+(y)), x >$$

for all $x \in X$ and $y \in D(T^+)$. Hence

$$T^*B = AT^+.$$

(d) The usual Hilbert space proof.

Theorem 4. *Let X and Y be continuous s.i.p.s. of type (p) and type (q), respectively, which are complete and uniformly convex and satisfy the homogeneity property. If T is a bounded linear operator from X into Y, then T^+ is bounded on Y and it holds that*

$$||T^+(y)|| \leq ||T||^{\frac{1}{p-1}} \cdot ||y||^{\frac{q-1}{p-1}}$$

for all $y \in Y$.

Proof. It A is well-known that the adjoint T^* of T is a bounded linear operator and $||T|| = ||T^*||$ holds. Then using Theorem 3. (a), (c) and Theorem 1. we obtain

$$||T^+(y)|| = ||(A^{-1}T^*B)(y)|| = ||T^*(B(y))||^{\frac{1}{p-1}}$$
$$\leq ||T||^{\frac{1}{p-1}} ||B(y)||^{\frac{1}{p-1}} == ||T||^{\frac{1}{p-1}} \cdot ||y||^{\frac{q-1}{p-1}}.$$

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Received by the editors October 27, 1991.