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ON A GENERALIZATION OF NEHARI'S CRITERION

Irinel Radomir, Nicolae N. Pascu
Department of Mathematics
"Transilvania" University
2200 Braşov, Romania

Abstract

There are many generalizations of Nehari's criterion for univalence (see [1]).

In this note we obtain a new generalization, using the theorem due to Ch. Pommerenke.

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1. Intoduction

In this note we obtain a new generalization, using the following theorem due to Ch. Pommerenke.

Let $U_r = \{z : z \in \mathbf{C}, |z| < r\}$, $r > 1$ and $U_1 = U$.

Theorem A. ([2]) *Let r_0 be a real number, $r_0 \in (0,1]$ and $U_{r_0} = \{z : |z| < r_0\}$. Let $f(z,t) = a_1(t)z + \ldots$, $a_1(t) \neq 0$, be regular for each $t \in I = [0,\infty)$ in $z \in U_{r_0}$ and locally absolutely continuous in I, locally uniform with respect to U_{r_0}. For almost all $t \in I$ suppose*

$$\frac{\partial}{\partial t} f(z,t) = z \frac{\partial}{\partial z} f(z,t) h(z,t),$$

$z \in U_{r_0}$, where $h(z,t)$ is regular in U and satisfies $\mathrm{Re}\, h(z,t) > 0$, *$z \in U$. If $|a_1(t)| \to \infty$ for $t \to \infty$ and if $f(z,t)/a_1(t)$ forms a normal family in U_{r_0}, then for each $t \in I$, $f(z,t)$ can be continued regularly in U, and gives univalent function.*

2. Main results

We denote by $\{f, z\}$,

$$\{f,z\} = (\frac{f''(z)}{f'(z)})' - \frac{1}{2}(\frac{f''(z)}{f'(z)})^2. \tag{1}$$

Theorem 1. *Let $f(z) = z + a_2 z^2 + \ldots$ be a regular function in U. If there exists a complex number k,* $\mathrm{Re}\, k \geq 1/2$, *such that*

$$|(1 - |z|^{2k})^2 \frac{z^2\{f,z\}}{2} + |z|^{2k}(1-k)| \leq |k||z|^{2Re\, k} \tag{2}$$

$(\forall) z \in U$, then the function $f(z)$ is univalent in U.

Proof. We consider the complex numbers $s = a + bi$ and $\sigma = \alpha + i\beta$, $a > 0$, $b, \beta \in \mathbf{R}$, $\alpha > 0$ and the function,

$$L(z,t) = f(e^{-st}z) + \frac{(e^{\sigma t} - e^{-st})zf'(e^{-st}z)}{1 - (e^{\sigma t} - e^{-st})zf''(e^{-st}z)/(2f'(e^{-st}z))}. \tag{3}$$

Because $f(z)$ is regular in U and $h(0,t) = 1$ for all $t \geq 0$, where

$$h(z,t) = 1 - \frac{(e^{\sigma t} - e^{-st})zf''(e^{-st}z)}{2f'(e^{-st}z)} \tag{4}$$

it results that there exists a real number $r_0 \in (0,1]$ such that $h(z,t) \neq 0$ for all $z \in U_{r_0}$ and $(\forall) t \geq 0$.

It results that the function $L(z,t)$ defined from (3) is regular in U_{r_0} for all $t \geq 0$.

Let us prove that the function $L(z,t)$ is a Loewner chain. From (3) it results that $L(z,t) = a_1(t)z + \ldots$, where $a_1(t) = e^{\sigma t} \neq 0$, for all $t \geq 0$ and

$$\lim_{t\to\infty} |a_1(t)| = \lim_{t\to\infty} e^{\alpha t} = \infty,$$

because $\alpha > 0$.

In order to prove that $L(z,t)/a_1(t)$ is a normal family in $U_{r_0}/2$ is sufficient to observe that by regularity of function $f(z)$, and because $a > 0$, $\alpha > 0$, there exist positive real numbers k_1, k_2 and k_3 such that

$$|e^{-\sigma t} f(e^{-st}z)| \le k_1,\ |(1 - e^{-(s+\sigma)t})zf'(e^{-st}z)| \le k_2 \tag{5}$$

and

$$|h(z,t)| \ge k_3,\ (\forall)z \in U_{r_0}/2 \text{ and } t \ge 0.$$

From (5) it results that

$$|L(z,t)/a_1(t)| \le k_1 + k_2/k_3,\ (\forall)z \in U_{r_0}/2$$

and $t \ge 0$ and hence, by Montel's theorem, it results that $L(z,t)/a_1(t)$ is a normal family in $U_{r_0}/2$.

To prove that $L(z,t)$ is a Loewner chain it is sufficient to prove that the function $p : U_{r_0}/2 \times I \to \mathbf{C}$, defined from

$$p(z,t) = \frac{z\frac{\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \tag{6}$$

has an analytic extension with positive real part in U for all $t \ge 0$. From (3) and (6) we have

$$p(z,t) = \frac{1 + e^{-(s+\sigma)t}\frac{(e^{\sigma t} - e^{-st})^2 z^2}{2}\{f, e^{-st}z\}}{-se^{-(s+\sigma)t}\frac{(e^{\sigma t} - e^{-st})^2 z^2}{2}\{f, e^{-st}z\}}. \tag{7}$$

We observe that the function has an analytic extension with positive real part in U for all $t \ge 0$ if and only if the function

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1} \tag{8}$$

has an analytic extension in U and $|w(z,t)| < 1$ for all $z \in U$ and $t \ge 0$. From (7) and (8) we obtain

$$w(z,t) = \frac{(1+s)e^{-(s+\sigma)t}\frac{(e^{\sigma t} - e^{-st})^2 z^2}{2}\{f, e^{-st}z\} + 1 - \sigma}{(1-s)e^{-(s+\sigma)t}\frac{(e^{\sigma t} - e^{-st})^2 z^2}{2}\{f, e^{-st}z\} + 1 + \sigma}. \tag{9}$$

If we consider the function

$$v(z,t) = e^{-(s+\sigma)t}\frac{(e^{\sigma t} - e^{-st})^2 z^2}{2}\{f, e^{-st}z\}, \tag{10}$$

regular in U for all $t \geq 0$, then we obtain

$$w(z,t) = \frac{(1+s)v(z,t) + 1 - \sigma}{(1-s)v(z,t) + 1 + \sigma} \tag{11}$$

Let $X = \operatorname{Re} v(z,t)$ and $Y = \operatorname{Im} v(z,t)$.

From (11) we obtain

$$|w|^2 = w\bar{w} = \frac{((1+s)v + 1 - \sigma)((1+\bar{s})v + 1 - \bar{\sigma})}{((1-s)v + 1 + \sigma)((1-\bar{s})v + 1 + \bar{\sigma})}$$

and hence the inequality $|w(z,t)| < 1$ is equivalent to the inequality

$$X^2 + Y^2 - \frac{\alpha - a}{a}X - \frac{\beta + b}{b}Y - \frac{\alpha}{a} < 0. \tag{12}$$

The inequality (12) can be written as

$$|v + 1 - \frac{\sigma + s}{2a}| \leq |\frac{\sigma + s}{2a}| \tag{13}$$

and inequality (13) is equivalent to the inequality $|w(z,t)| < 1$.

If $k = (\sigma + s)/(2a)$, from hypothesis, $\operatorname{Re} k \geq 1/2$, and hence

$$|k - 1| \leq |k|. \tag{14}$$

From (13) and (14) we have

$$|v(z,0) + 1 - \frac{\sigma + s}{2a}| \leq |\frac{\sigma + s}{2a}| \text{ for all } z \in U. \tag{15}$$

By maximum principle, the results from (15) that inequality (13) holds true for all $z \in U$ and $t = 0$. We observe that for $z = 0$ and $t > 0$, $v(0,t) = 0$ and, hence, from $\operatorname{Re} k \geq 1/2$ and from maximum principle, inequality (13) holds.

For all $t > 0$, $z \in U$, $z \neq 0$ we observe that the function $v(z,t)$ is regular in $\bar{U} = \{z : |z| \leq 1\}$ and hence

$$(16) \qquad |v(z,t) + 1 - \frac{\sigma + s}{2a}| < \max_{|z|=1} |v(z,t) + 1 - \frac{\sigma + s}{2a}| = |v(e^{i\theta}, t) + 1 - \frac{\sigma + s}{2a}|,$$

where θ is a real number. If $u = e^{-st}e^{i\theta}$, then $|u| = e^{-st}$ and hence,

$$v(e^{i\theta}, t) = |(1 - |u|^{2k})^2 \frac{u^2\{f, u\}}{2} + |u|^{2k}(1 - k)|/|u|^{2k}.$$

Because $|u| = e^{-st}$ and $a > 0$ it results that $u \in U$ and hence from (2), we obtain

$$|((1 - |u|^{2k})^2 \frac{u^2\{f, u\}}{2} + |u|^{2k}(1 - k))/|u|^{2k} + (1 - k)| \leq |k|$$

and hence

$$(17) \qquad |v(e^{i\theta}, t) + 1 - \frac{\sigma + s}{2a}| \leq |k|.$$

From (16) and (17) results that inequality (13) holds for all $z \in U$, $z \neq 0$ and $t \geq 0$.

It results that $L(z,t)$ is a Loewner chain, and hence the function $L(z,0) = f(z)$ is univalent in U.

Observation. For $k = 1$, from Theorem 1 we obtain Nehari's criterion for univalence.

References

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