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JACOBI POLYNOMIALS IN SPECTRAL APPROXIMATION FOR SHOCK LAYER PROBLEMS

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Abstract

The shock layer problem, described by the second order linear differential equation, with the single turning point is considered. The solution is presented as a sum of the left and right reduced solution and the layer function, which is approximated by the truncated Jacobi orthogonal series. The layer subinterval is determined through the numerical layer length, which depends on the perturbation parameter and the degree of the spectral approximation. The coefficients in the differential equation are approximated by the appropriate power series and certain recurrence relations between the coefficients in the Jacobi orthogonal series are presented. The upper bound for the error function is constructed and the numerical example is included.

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1. Introduction

In this paper we shall consider the single turning point problem

(1) $$Ly(x) = -\varepsilon^2 y''(x) - xf(x)y'(x) + g(x)y(x) = 0, x \in [-1, 1]$$

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$$y(-1) = A, y(1) = B, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter, A, $B \in R$, $A \neq B$, $f(x)$, $g(x) \in C[-1,1]$ and

$$f(x) \geq \alpha > 0,\ \alpha \in R,\ \ell = \frac{g(0)}{f(0)} = 0. \tag{2}$$

It is well known (see e.g. [6]) that under the assumptions (2) the problem (1), (1.2) has the so-called "shock layer" at the point $x = 0$. The problems of this kind are involved in mathematical models of diffusion-convection phenomenon and it has been recognized that certain difficulties arise when standard spectral approximations are applied in the cases where ε is very small. The author has already developed the modification of the standard spectral methods for the singularly perturbed problems without turning points (see e.g. [1]) and for turning point problems with boundary layers (see e.g. [2]), where some investigations for the shock layer problems were also carried out. In that paper the author developed the modification of standard τ-method in detail for the single turning point problems without interior layer, and some ideas for the application of the similar method for the shock layer problems. In this paper the author developes these ideas. The modification is combined with the use of Jacobi orthogonal basis, so that certain recurrence relations, obtained by the author in [1], can be applied to obtain a highly accurat method using only a small number of terms in the appropriate truncated series. This is confirmed by the numerical results. The use of Jacobi orthogonal basis represents the generalization which enables the application of a large number of well known classis of classical orthogonal polynomials such as Legendre or Chebyshev basis.

In Section 2 the original problem will be transformed and the layer subintervals will be determined, using the same idea as in the case of nonselfadjoint problems, but a different technique is requested. In Section 3 the orthogonal projecting, according to Jacobi basis, will be defined and certain recurrence relations will be developed. In Section 4 the upper bound function for the error estimate will be constructed, and in Section 5 the theoretical results will be illustrated by a numerical example.

2. Transformation of the problem

In [6] it was shown that under the asumptions (2) we have

$$(3) \qquad y(x) = \begin{cases} y_L(x) + 0(\varepsilon^2) & -1 \le x < 0 \\ 0(1) & x = 0 \\ y_R(x) + 0(\varepsilon^2) & 0 < x \le 1 \end{cases},$$

where $y_L(x)$ and $y_R(x)$ are left and right reduced solutions, which satisfy

$$(4) \qquad -xf(x)y_L'(x) + g(x)y_L(x) = 0,\ x \in [-1,0),\ y_L(-1) = A$$

$$(4.2) \qquad -xf(x)y_R'(x) + g(x)y_R(x) = 0,\ x \in (0,1],\ y_R(1) = B,$$

and

$$(4.3) \qquad y(0) \to \frac{1}{2}(y(-1) + y(1)) = \frac{1}{2}(y_L(0) + y_R(0)).$$

We are going to look for the solution of (1), (1.2) in the form

$$(5) \qquad y(x) = \begin{cases} y_L(x) + u_\varepsilon(x) & x \in [-1,0) \\ y_R(x) + w_\varepsilon(x) & x \in (0,1] \end{cases},$$

where $y_L(x)$ and $y_R(x)$ are determined by (4), (4.2) and $u_\varepsilon(x)$ and $w_\varepsilon(x)$ represent the solutions of the boundary value problems:

$$(6) \quad Lu_\varepsilon(x) = \varepsilon^2 y_L''(x), x \in [-1,0), u_\varepsilon(-1) = 0, u_\varepsilon(0) = A^0 = \frac{B-A}{2},$$

$$(6.2) \qquad Lw_\varepsilon(x) = \varepsilon^2 y_R''(x), x \in (0,1], w_\varepsilon(0) = B^0 = \frac{A-B}{2}, w_\varepsilon(1) = 0.$$

The boundary conditions in (6) and (6.2) are determined by the use of (4.3).

The idea is to approximate $u_\varepsilon(x)$ and $w_\varepsilon(x)$ by

$$(7) \quad u(x) = \begin{cases} 0 & x \in [-1,-\delta] \\ z(x) & x \in [-\delta,0) \end{cases} \text{ and } w(x) = \begin{cases} v(x) & x \in (0,\delta] \\ 0 & x \in [\delta,1] \end{cases},$$

where $z(x)$ and $v(x)$ satisfy

$$(8) \qquad Lz(x) = \varepsilon^2 y_L''(x), x \in [-\delta,0), z(-\delta) = 0, z(0) = A^0,$$

$$Lv(x) = \varepsilon^2 y_R''(x), x \in (0, \delta], v(0) = B^0, v(\delta) = 0. \tag{8.2}$$

All further investigations will be carried out for the interval $[0, 1]$.

The division point $\delta > 0$ is the so-called numerical layer length which is going to be determined using the following definitions and lemmas

Definition 1. *A function $p(x) \in C^2(0, \delta]$ is called a resemblance function for the problem (8.2) if*

1. *it satisfies the boundary conditions in (8.2).*
2. *$x = \delta$ is the stationary point for $p(x)$*
3. *$p(x)$ is concave for $B^0 > 0$ and convex for $B^0 < 0$.*

Lemma 1. *The n-th degree polynomial*

$$p_n(x) = a\left(\frac{\delta - x}{\delta}\right)^n, a = \frac{1}{2}(y_L(0) - y_R(0)), n \geq 2 \tag{9}$$

is a resemblance function for the problem (8.2).

Proof. We have to verify the conditions from Def. 1.

1. $p_n(0) = a = \frac{1}{2}(y_L(0) - y_R(0)) = B^0 = v(0)$ (by the use of (4.3) and the boundary condition in (6.2)) and $p_n(\delta) = 0 = v(\delta)$.
2. $p_n'(x) = 0$ only for $x = \delta$.
3. $\mathrm{sgn} p_n''(x) = \mathrm{sgn} B^0$.

Definition 2. *The sufficiently small number δ for which the resemblance function satisfies the differential equation in (8.2) in the neighbourhood of the layer point, $x = 0$ is called the numerical layer length.*

Lemma 2. *The numerical layer length for the problem (8.2) is*

$$\delta \approx \varepsilon\sqrt{\frac{n(n-1)}{f(0)}}. \tag{10}$$

Proof. By substituting (9) into (8.2) we obtain

$$-\varepsilon^2\frac{an(n-1)}{\delta^2}(\frac{\delta-x}{\delta})^{n-2}+xf(x)\frac{an}{\delta}(\frac{\delta-x}{\delta})^{n-1}+g(x)a(\frac{\delta-x}{\delta})^n=\varepsilon^2 y_R''(x).$$

At the neighbourhood point $x=\frac{\delta}{n}$ of the layer point $x=0$ this will give

$$-\varepsilon^2\frac{an(n-1)}{\delta^2}(1-\frac{1}{n})^{n-2}+f(\frac{\delta}{n})a(1-\frac{1}{n})^{n-1}+g(\frac{\delta}{n})a(1-\frac{1}{n})^n=\varepsilon^2 y_R''(\frac{\delta}{n}).$$

For a sufficiently small δ and sufficiently large n $f\left(\frac{\delta}{n}\right)\approx f(0)$, $g\left(\frac{\delta}{n}\right)\approx g(0)=0$ and $\left(1-\frac{1}{n}\right)^{n-j}\approx\frac{1}{e}$ for $j=0,1,2$, so that the above equation becomes

$$-\varepsilon^2\cdot\frac{an(n-1)}{\delta^2}+f(0)a=\varepsilon^2 e y_R''(0).$$

This gives

$$\delta=\varepsilon\sqrt{\frac{an(n-1)}{af(0)-\varepsilon^2 e y_R''(0)}}\approx\varepsilon\sqrt{\frac{n(n-1)}{f(0)}},$$

when ε is small.

Now we can proceed to construct the approximate solution fot he problem (8.2), using Jacobi polynomial basis. In that purpose we must, first, transform the interval $[0,\delta]$ into $[-1,1]$ using the substitution $x=\frac{\delta}{2}(t+1)$. Thus, (8.2) becomes

(11) $\quad L_\delta V(t)\equiv-\mu^2V''(t)-F(t)V'(t)+G(t)V(t)=H(t),\ t\in[-1,1]$

(11.2) $$V(-1)=B^0,\ V(1)=0,$$

with

(11.3) $$V(t)=v\left(\tfrac{\delta}{2}(t+1)\right),\mu=\tfrac{2\varepsilon}{\delta},\quad F(t)=(t+1)f\left(\tfrac{\delta}{2}(t+1)\right),$$
$$G(t)=G\left(\tfrac{\delta}{2}(t+1)\right),\qquad H(t)=\varepsilon^2 y_R''\left(\tfrac{\delta}{2}(t+1)\right).$$

3. Jacobi spectral approximation

The spectral solution of the problem (11), (11.2) will be represented as a truncated orthogonal series of degree n, according to Jacobi basis of the space P_n of all real polynomials of degree up to n. Let, first, remind us of their properties.

Classical Jacobi polynomials $P_k^{\alpha,\beta}(t)$, $\alpha > -1$, $\beta > -1$, represent a particular solution of the differential equation

$$(1-t^2)\phi''(t) + (\beta - \alpha - (\alpha+\beta+2)t)\phi'(t) + k(k+\alpha+\beta+1)\phi(t) = 0,$$
$$t \in [-1,1], k \in N_0$$

and they satisfy the recurrence Bonnet's relation

$$P_{k+1}^{\alpha,\beta}(t) - (\alpha_k t + \beta_k)P_k^{\alpha,\beta}(t) + \gamma_k P_{k-1}^{\alpha,\beta}(t) = 0, k = 0,1,\dots \tag{12}$$
$$P_0^{\alpha,\beta}(t) = 1, \qquad P_{-1}^{\alpha,\beta}(t) = 0,$$

with

$$\alpha_k = \frac{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}{2(k+1)(k+\alpha+\beta+1)}, \tag{12.2}$$

$$\beta_k = \frac{(2k+\alpha+\beta+1)(\alpha^2-\beta^2)}{2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)},$$

$$\gamma_k = \frac{(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)}{(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)}.$$

For the derivatives we have

$$(1-t^2)(P_k^{\alpha,\beta}(t))' = (u_k t + v_k)P_k^{\alpha,\beta}(t) - w_k P_{k-1}^{\alpha,\beta}(t) \tag{13}$$

with

$$u_k = -k,\ v_k = \frac{k(\alpha-\beta)}{2k+\alpha+\beta}, \quad w_k = \frac{-2(k+\alpha)(k+\beta)}{2k+\alpha+\beta}. \tag{13.2}$$

When speaking of Jacobi spectral approximation of the solution $V(t)$ of the problem (11), (11.2), we, in fact, consider a truncated orthogonal series

$$V_n(t) = \sum_{k=0}^{n} a_k P_k^{\alpha,\beta}(t), \tag{14}$$

such that

$$L_\delta V_n(t) = H(t),\ t \in [-1,1],\ V_n(-1) = B^0,\ V_n(1) = 0. \tag{15}$$

It is well known that if $n \to \infty$ then $V_n(t) \to V(t)$.

Here, we have to overcome two difficulties: First, to express $(P_k^{\alpha,\beta}(t))'$ and $(P_k^{\alpha,\beta}(t))''$ through $P_k^{\alpha,\beta}(t)$, and to represent products $F(t)V_n'(t)$ and $G(t)V_n(t)$ in the form of truncated Jacobi orthogonal series. To that purpose we whall approximate $F(t)$ and $G(t)$ by the power series

$$F(t) = \sum_{j=0}^{n} f_j t^j \quad \text{and} \quad G(t) = \sum_{j=0}^{n} g_j t^j. \tag{16}$$

Further we shall need some results obtained by the author in [1]:

a) Starting from (12) and multiplying it by t $(j-1)$ times, and using the following notation each time, we can easily see that

$$t^j P_k^{\alpha,\beta}(t) = \sum_{i=k-j}^{k+j} A_i^j P_i^{\alpha,\beta}(t), \tag{17}$$

with

$$A_i^j = \frac{A_{i-1}^{j-1}}{\alpha_{i-1}} - \frac{\beta_i A_i^{j-1}}{\alpha_i} + \frac{\gamma_i A_{i+1}^{j-1}}{\alpha_{i+1}}, \; i = k-j, \dots, k+j, \tag{17.2}$$

$$A_{i+1}^1 = \tfrac{1}{\alpha_i}, \; A_{A_i}^1 = \tfrac{-\beta_i}{\alpha_i}, \; A_{i-1}^1 = \tfrac{\gamma_i}{\alpha_i}, \; i \in N_0, \tag{17.3}$$

where for $i = k-j+1$ the first term is omitted, for $i = k+j-1$ the last one, for $i = k-j$ the first two, and for $i = k+j$ the last two terms are omitted.

b) The first derivative of $P_k^{\alpha,\beta}(t)$, being a polynomial of $k-1$ degree, can be represented exactly as a linear combination of the Jacobi basis of space P_{k-1}, i.e.

$$(P_k^{\alpha,\beta}(t))' = \sum_{i=0}^{k-1} b_i^{(1)} P_i^{\alpha,\beta}(t). \tag{18}$$

Introducing (18) into (13) and making use of (17) for $j = 2$, we come to the following recurrence relations:

$$b_{k-1}^{(1)} = \frac{-u_k A_{k+1}^1}{A_{k+1}^2}, \quad b_{k-2}^{(1)} = \frac{-u_k A_k^1}{A_k^2} - \frac{v_k}{A_k^2} - b_{k-1}^{(1)}$$

$$(19)\quad b^{(1)}_{k-3} = \frac{-u_k A^1_{k-1}}{A^2_{k-1}} + \frac{w_k}{A^2_{k-1}} - b^{(1)}_{k-2} - b^{(1)}_{k-1}\left(1 - \frac{1}{A^2_{k-1}}\right)$$

$$b^{(1)}_{i-2} = b^{(2)}_{i-1} - b^{(1)}_i\left(1 - \frac{1}{A^2_i}\right) - b^{(1)}_{i+1} - b^{(1)}_{i+2}, \quad i = k-2, \ldots, 2.$$

Using the similar technique for the second derivative we obtain

$$(20)\qquad (P^{\alpha,\beta}_k(T))'' = \sum_{i=0}^{k-2} b^{(2)}_i P^{\alpha,\beta}_i(t)$$

where

$$b^{(2)}_{k-2} = k(k+\alpha+\beta+1) - (\alpha+\beta+2)b^{(1)}_{k-1},\ b^{(2)}_{k-3} =$$

$$= -b^{(2)}_{k-2} + b^{(2)}_{k-1}(-(\alpha+\beta+2)\frac{A^1_{k-1}}{A^2_{k-1}} + \frac{\beta-\alpha}{A^2_{k-1}}) - b^{(1)}_{k-1}\frac{(\alpha+\beta+2)}{A^2_{k-1}}$$

$$(20.2)\quad b^{(2)}_{i-2} = b^{(2)}_{i-1} + b^{(2)}_i(\frac{1}{A^2_i} - 1) - b^{(2)}_{i+1} - b^{(2)}_{i+2} - b^{(1)}_{i+1}(\alpha+\beta+2)\frac{A^1_i}{A^2_i} +$$

$$b^{(2)}_i\frac{\beta - \alpha(\alpha+\beta+2)A^1_i}{A^2_i} - b^{(1)}_{i-1}\frac{(\alpha+\beta+2)A^1_i}{A^2_i} \quad i = k-2, \ldots, 2.$$

Finally, if we represent the function $H(t)$ as

$$(21)\qquad H(t) = \sum_{k=0}^{n} h_k P^{\alpha,\beta}_k(t),$$

we can prove the following theorem:

Theorem 1. *The coefficients a_k in the solution (14) of the problem (11), (11.2), obtained using τ-method, represent the solution of the system*

$$(22)\quad -\mu^2 b^{(2)}_i \sum_{k=i+2}^{n} a_k + \sum_{j=0}^{n} A^j_i(f_j \sum_{r=M}^{m}(b^{(1)}_r \sum_{k=r+1}^{n} a_k) + g_j \sum_{k=M}^{m} a_k) = h_i$$

$$i = 0, \ldots, n-2$$

$$(22.2)\qquad \sum_{k=0}^{n}(-1)^k\binom{k+\alpha}{k}a_k = B^0, \qquad \sum_{k=0}^{n}\binom{k+\alpha}{k}a_k = 0,$$

where $M = \max(0, i-j)$, $m = \min(n, i+j)$.

Proof. In order to obtain the equations (22) we subsitutute (14) and (21) into (15) and make use of (18), (20) and (17). After equating the coefficients at $P_i^{\alpha,\beta}(t)$, $i = 0, \ldots, n-2$ we come to (22). The equations (22.2) are obtained directly from the boundary conditions in (15), using that

$$P_k^{\alpha,\beta}(1) = \binom{k+\alpha}{k} \text{ and } P_k^{\alpha,\beta}(-t) = (-1)^k P_k^{\alpha,\beta}(t), \quad k = 0, 1, \ldots$$

After the system (22), (22.2) is solved for a_k, $k = 0, \ldots, n$, we obtain the approximate solution

$$y_n(x) = y_R(x) + v_n(x) = y_R(x) + V_n(\frac{2x}{\delta} - 1), \quad x \in (0, \delta]. \tag{23}$$

4. The error estimate

Out of the boundary layer, the exact solution of the problem (1), (1.2) is approximated by the solution of the reduced problem. According to (3) the following estimate is valid

$$d(x) = |y(x) - y_R(x)| \leq C\varepsilon^2, \quad x \in [\delta, 1]. \tag{24}$$

Throughout the paper C will denote an arbitrary constant independent of x and ε.

Let us now estimate the error upon the layer subinterval $(0, \delta]$. The error function, according to (5) and (23) is

$$d(x) = |y(x) - y_n(x)| = |w_\varepsilon(x) - v_n(x)|. \tag{25}$$

In order to estimate it, we have, first, to prove the following lemma:

Lemma 3. *Let $g(x) \geq K^2$, $K \in R$ for $x \in (0, \delta]$. Then*

$$|w_\varepsilon(x) - v(x)| \leq C\varepsilon^2 \text{ for } x \in (0, \delta]. \tag{26}$$

Proof. The function $w_\varepsilon(x)$ satisfies the boundary value problem

$$Lw_\varepsilon(x) = \varepsilon^2 y_R''(x),\ x \in (0, \delta],\ w_\varepsilon(0) = B^0,\ w_\varepsilon(\delta) = y(\delta) - y_R(\delta). \tag{27}$$

Subtracting (8.2) from (27) we obtain

$$L(w_\varepsilon - v)(x) = 0,\ (w_\varepsilon - v)(0) = 0,\ (w_\varepsilon - v)(\delta) = y(\delta) - y_R(\delta).$$

As $g(x) \geq K^2$ and, according to (2), $xf(x) > 0$, for $x \in (0,\delta]$ the operator L is inverse monotone. So, by the principle of inverse monotonicity we can conclude that

(28) $$|w_\varepsilon(x) - v(x)| \leq |y(\delta) - y_R(\delta)|.$$

Using the estimate (24) for $x = \delta$ we obtain (26).

Theorem 2. *Let ω_i, $i = 1,2$ be the exact solutions of the problems*

(29) $$-\varepsilon^2\omega_1''(x) - xF_1\omega_1'(x) = 0\ x \in (0,\delta],\ \omega_1(0) = B^0,\ \omega_1(\delta) = 0$$

(29.2) $$-\varepsilon^2\omega_2''(x) - \delta F_2\omega_2'(x) + K_2^2\omega_2(x) = 0, x \in (0,\delta], \omega_2(0) = B^0, \omega_2(\delta) = 0,$$

where F_1, F_2, K_2, $K \in R$ are such constants that $F_1 \leq f(x) \leq F_2$, $K^2 \leq g(x) \leq K_2^2$ while $x \in (0,\delta]$ and

(30) $$d_n(x) = \max_i\{|\omega_i(x) - v_n(x)|\}.$$

Then the error $d(x)$, defined by (25) can be estimated as

(31) $$d(x) \leq C\varepsilon^2 + d_n(x), \quad x \in (0,\delta].$$

Proof. We can see that

(32) $$d(x) \leq |w_\varepsilon(x) - v(x)| + |v(x) - v_n(x)|,$$

where $v(x)$ is the solution of the problem (8.2).

Let us, first, assume that $B^0 > 0$. Then, applying the principle of inverse monotonicity to the problems (29) and (29.2), we have

(33) $$\omega_i(x) \geq 0 \text{ and } \omega_i'(x) \leq 0,\ i = 1,2 \quad \text{for } x \in (0,\delta].$$

Defining the functions $f_i(x) \geq 0$, $i = 1,2,3$ and $g_2(x) \geq 0$, such that $f(x) = F_1 + f_1(x)$, $f(x) = F_2 - f_2(x)$, $F_2 = F_1 + f_3(x)$, $g(x) = K_2^2 - g_2(x)$ for $x \in (0,\delta]$, using (29), (29.2) we obtain

$$\begin{aligned} L\omega_2(x) &= -\varepsilon^2\omega_2'' - x(F_2 - f_2(x))\omega_2' + (K_2^2 - g_2(x))\omega_2 = \\ &= (\delta - x)F_2\omega_2' + xF_2\omega_2' - g_2\omega_2 \end{aligned}$$

and

$$L\omega_1(x) = -\varepsilon^2\omega_1'' - x(F_1 + f_1(x))\omega_1' + g(x)\omega_1 = -xf_1(x)\omega_1' + g(x)\omega_1,$$

i.e., according to (2) and (33) we have

$$L\omega_1(x) \geq 0 \quad \text{and} \quad L\omega_2(x) \leq 0. \tag{34}$$

We can also, see that

$$\begin{aligned}&-\varepsilon^2(\omega_2-\omega_1)'' - \delta F_2(\omega_2-\omega_1)' + K_2^2(\omega_2-\omega_1) = \\ &= xf_3(x)\omega_1' + (\delta - x)F_2\omega_1' - K_2^2\omega_1 < 0,\ x \in (0,\delta], \\ &(\omega_2-\omega_1)(0) = (\omega_2-\omega_1)(\delta) = 0,\end{aligned}$$

which, using the principle of inverse monotonicity, gives that

$$(\omega_2-\omega_1)(x) \leq 0,\ \text{i.e. } \omega_2(x) \leq \omega_1(x) \text{ for } x \in (0,\delta]. \tag{35}$$

Applying the result given by Lorenz in [5], the relations (34), (35) and boundary conditions in (29), (29.2) give the following inequality

$$\omega_2(x) \leq v(x) \leq \omega_1(x), \quad x \in (0,\delta],$$

where $v(x)$ is the solution of the problem (8.2).

After subtracting $v_n(x)$ in the above inequality we conclude that

$$|v(x) - v_n(x)| \leq \max_i\{|\omega_i(x) - v_n(x)|\} = d_n(x) \tag{36}$$

In the case $B^0 < 0$, using the same technique, we, again, come to (36). Finally, using the estimates (26) and (36) in (32) we obtain (31).

5. Numerical example

We shall use the following test example:

$$-\varepsilon^2 y''(x) - xy'(x) = 0 \quad x \in [-1,1], y(-1) = 0, y(1) = 2,$$

with the solution

$$y(x) = 1 + \frac{erf(x/(\sqrt{2}\varepsilon))}{erf(1/(\sqrt{2}\varepsilon))}.$$

The left reduced solution is $y_L(x) \equiv 0$, and the right reduced solution is $y_R(x) \equiv 2$. The numerical layer length, applying (10), is

$$\delta = \varepsilon\sqrt{n(n-1)}.$$

In the following table we shall give the values of the exact solution and the error $d(x)$ in several points from the layer for $x > 0$. The results for $x < 0$ are the same. The results are obtained for $\alpha = \beta = 0$ i.e. Legendre basis

Table 1.

$\varepsilon = 10^{-7}$		$n = 8$	$n = 12$
x	$y(x)$	$d(x)$	$d(x)$
0,00000001	1.08	$1.3 \cdot 10^{-3}$	$4.9 \cdot 10^{-4}$
0,00000005	1.38	$5.9 \cdot 10^{-4}$	$3.0 \cdot 10^{-3}$
0,00000008	1.58	$7.5 \cdot 10^{-5}$	$2.2 \cdot 10^{-3}$
0,0000001	1.68	$3.0 \cdot 10^{-4}$	$2.3 \cdot 10^{-4}$
0,0000002	1.95	$2.2 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$

Remark. The above results are of the same order as the results obtained in [2] where the Chebyshev basis combined with the collocation method was used.

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