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A DIFFERENCE SCHEME FOR BOUNDARY PROBLEM WITH TURNING POINT

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Abstract

In [7] a family of spline difference scheme is derived. The El Mistikawy and Werle scheme (EMW scheme) is a member of the family. The applying of that scheme on the problem with turning point is considered in [2]. The application of another member of the family (IEMW scheme) giving a better accuracy on turning point problems is analysed in [8]. Here we consider the solving of turning point problems using a genaralized scheme from [9]. The numerical results show a better accuracy for that scheme than for the both mentioned schemes. The theoretical result is not sharp for fixed ε.

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1. Introduction

Let us consider the problem

$$(1)\qquad \begin{cases} Ly = \varepsilon y'' + p(x)y' - q(x)y = f(x), \quad x \in I = [-1,1], \\ y(-1) = \alpha_0, \quad y(1) = \alpha_1, \end{cases}$$

where ε is a small positive parameter, α_0 and α_1 are given numbers, $p(x)$, $q(x)$ and $f(x)$ are sufficiently smooth functions and $q(x) \geq q > 0$, $q \in R$. Further, we assume that: $p(x)$ has a simple zero at $x = 0$ and no other zeros on $[-1,1]$, $|p'(x)| \geq |p'(0)/2|, x \in I$. Let $\beta = q(0)/p'(0) \geq 0$, and let β_l, β_s be fixed constants such that $\beta_l \leq 1 \leq \beta_s$ and $\beta_l \leq |\beta| \leq \beta_s$. For numerical solving the problem (1) we derive a scheme by using the linear combination of the two spline difference scheme: El Mistikawy Werle (EMW) from [2] and Improved El MIstikawy and Werle (IEMW) scheme from [7] or [8]. The linear combination is expressed via the pa rameter λ and the scheme has the form:

$$(2) \qquad \begin{cases} \varepsilon h^{-2} R u_j = Q f_j, \quad j = 1(1)n \\ u_0 = \alpha_0, \quad u_1 = \alpha_1, \end{cases}$$

where

$$R u_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1}$$

$$Q f_j = \lambda q_j^- f_{j-1} + (1-\lambda) q_j^{\mp} f_{j-1/2} + \lambda q_j^c f_j + (1-\lambda) q_j^{\pm} f_{j+1/2} + (1-\lambda) q_j^+ f_{j+1}$$

and

$$r_j^- = \lambda \tilde{r}_a + (1-\lambda)\dot{r}_a, \quad r_j^+ = \lambda \tilde{r}_b + (1-\lambda)\dot{r}_b,$$

$$r_j^c = \lambda \tilde{r}_c + (1-\lambda)\dot{r}_c + \lambda \tilde{r}_d + (1-\lambda)\dot{r}_d,$$

$$q_j^- = e(\tilde{k}_1, \tilde{n}_1), \quad q_j^+ = s(\tilde{k}_2, \tilde{n}_2), \quad q_j^c = q_j^- + q_j^+$$

$$q_j^{\mp} = 2e(\dot{k}_1, \dot{n}_1), \quad q_j^{\pm} = 2s(\dot{k}_2, \dot{n}_2),$$

$$a(k,n) = (n-k)exp(n)/(1-exp(n-k)), b(k,n) = (n-k)exp(-k)/(1-exp(n-k)),$$

$$c(k,n) = -n - 1/g(n-k), \quad d(k,n) = k - 1/g(n-k),$$

$$g(u) = (exp(u) - 1)/u, g(0) = 1,$$

$$e(k,n) = (g(n) - exp(n)g(-k))/(2 - 2exp(n-k)),$$

$$s(k,n) = (g(-k) - exp(-k)g(n))/(2 - 2exp(n-k)),$$

$$\tilde{r}_a = a(\tilde{k}_1, \tilde{n}_1), \dot{r}_a = a(\dot{k}_1, \dot{n}_1),$$

$$\tilde{r_b} = b(\tilde{k}_2, \tilde{n}_2), \quad \dot{r_b} = b(\dot{k}_2, \dot{n}_2),$$
$$\tilde{r_c} = c(\tilde{k}_1, \tilde{n}_1), \quad \dot{r_c} = c(\dot{k}_1, \dot{n}_1),$$

$$\tilde{r_d} = d(\tilde{k_2}, \tilde{n_2}), \dot{r_d} = d(\dot{k}_2, \dot{n}_2),$$

$$\tilde{n}_1 = h\tilde{a}_1, \tilde{k}_1 = h\tilde{b}_1, \tilde{n}_2 = h\tilde{a}_2, \tilde{k}_2 = h\tilde{b_2}$$

and $\tilde{a_1}, \tilde{b_1}, \tilde{a_2}, \tilde{b_2}$ $(\tilde{a_1} < \tilde{b_1}, \tilde{a_2} < \tilde{b_2})$ are solutions of the equations

$$\varepsilon w^2 + \tilde{p}_j w - \tilde{q}_j = 0, \quad \varepsilon w^2 + \tilde{p}_{j+1} w - \tilde{q}_{j+1} = 0,$$

respectively. Further,

$$\dot{n_1} = h\dot{a}_1, \dot{k}_1 = h\dot{b}_1, \dot{n}_2 = h\dot{a}_2, \dot{k}_2 = h\dot{b_2}$$

and $\dot{a}_1, \dot{b}_1, \dot{a}_2, \dot{b}_2$ $(\dot{a}_1 < \dot{b}_1, \dot{a}_2 < \dot{b}_2)$ are solutions of the equations

$$\varepsilon w^2 + \dot{p}_j w - \dot{q}_j = 0, \quad \varepsilon w^2 + \dot{p}_{j+1} w - \dot{q}_{j+1} = 0,$$

respectively. Here

$$\tilde{F}_j = (F_{j-1} + F_j)/2, \quad \dot{F}_j = F_{j-1/2}.$$

When $\lambda = 1$ we obtain the EMW scheme or the scheme from [2], when $\lambda = 0$ and $q = 0$ we obtain the IEMW scheme, or for $q \neq 0$ we have the scheme from [8] and for $\lambda = -1/3$ and $q = 0$ we obtain the scheme from[9].

The modifications are made in a similar way as a modification of the EMW scheme in [2].

Namely, in order to follow the estimates of exact solution from [2], we require that $\tilde{p} \leq M|p(x)|$, and $\dot{p} \leq M|p(x)|$, $x \in X'$, $X' = \bigcup_{j=1}^{n+1}(x_{j-1}, x_j)$, M is a constant independent of ε and h. This condition may be satisfied by modifying the choice of $\tilde{p}_i$ or $\dot{p}_i$ near the turning point (like in [2]): if there is a mesh point x_i coinciding with the turning point $x = 0$, then we put $\tilde{p}_i = 0$ for that index i. If the turning point is in the interior of (x_i, x_{i+1}), then we put $\tilde{p}_i = p(x_i)$, $\tilde{p}_{i+1} = 0$ and $\tilde{p}_{i+2} = p(x_{i+1})$ for $x \in (x_{i+1}, x_{i+2})$. In a similar way we modify the approximation $\dot{p}_i$. Now, we can prove the following theorem.

Theorem 1. *Let the above conditions on functions $p(x), q(x)$ and $f(x)$ are fulfilled. Let $p \in C^2[-1,1]$, q and $f \in C^1[-1,1]$. Then, there are the constants M_1, M_2 and m independent of ε and h such that:*

$$|y(x_j) - u_j| \leq M_2(h^\beta + h) \quad for \beta \neq 1$$

$$|y(x_j) - u_j| \leq M_1 h \quad ln(1/(mh^2) \quad for \beta = 1$$

where u_j is the approximation to $y(x_j)$ at the point x_j.

The proof follows from the conection $\tilde{p_j} = \dot{p}_j + O(h^2)$ and the proof from [2] for the modified EMW scheme. The better numerical results for the linear combination are a consequence of the fact that this scheme has the fourth order of classical convergence for the singular problem without a turning point, (see [9]).

2. Numerical results

In this section we present the results of some numerical experiments using the EMW, IEMW schemes and our new scheme. We denote by E_n the maximum of $|y(x_j) - u_j|, j = 0(1)n + 1$. Here $[u_0, u_1, \ldots, u_{n+1}]^\top$ is the corresponding numerical solution to the system (2). Different values of $\varepsilon = 2^k$ and n are considered.

The example is taken from [2], and the exact solution is modifed by adding x^2. Some explanation for that, one can find in [7]¿

Example 1.

$$\begin{cases} p(x) = ((1/2 - x) - 3.121(x - 1/2)^2)/\beta, \\ q(x) = -1 - .2764(x - 1/2), \\ y(x) = (.291(x - z)^2 + \varepsilon)^{\beta/2} + (x - z)^2 + \varepsilon)^{(\beta-1)/2} + \exp(-.5x^2) + x^2 \end{cases}$$

k	n							
	8	16	32	64	128	256	512	
1	4.18(−3)	1.05(−3)	2.62(−4)	6.56(−5)	1.64(−5)	4.10(−6)	1.02(−6)	E_n
2	5.83(−3)	1.48(−3)	3.70(−4)	9.27(−5)	2.32(−5)	5.79(−6)	1.45(−6)	E_n
3	8.50(−3)	2.16(−3)	5.43(−4)	1.36(−4)	3.40(−5)	8.50(−6)	2.13(−6)	E_n
4	1.33(−2)	3.45(−3)	8.70(−4)	2.19(−4)	5.48(−5)	1.37(−5)	3.43(−6)	E_n
5	1.92(−2)	5.27(−3)	1.42(−3)	3.58(−4)	8.95(−5)	2.24(−5)	5.61(−6)	E_n
6	2.24(−2)	9.03(−3)	2.36(−3)	5.98(−4)	1.50(−5)	3.76(−5)	9.41(−6)	E_n
7	1.83 (−2)	1.43(−2)	3.83(−3)	1.04(−4)	2.62(−4)	6.56(−5)	1.64(−6)	E_n
8	1.09(−2)	1.78(−2)	7.04(−3)	1.83(−3)	4.65(−4)	1.18(−4)	2.94(−5)	E_n
9	6.62(−3)	1.69(−2)	1.18(−2)	3.14(−3)	8.46(−4)	2.13(−4)	5.34(−5)	E_n
10	9.62(−3)	1.28(−2)	1.49(−2)	5.86(−3)	1.52(−3)	3.86(−4)	9.78(−5)	E_n
11	2.69(−2)	9.27(−3)	1.51(−2)	9.92(−3)	2.65(−3)	7.11(−4)	1.79(−5)	E_n
12	4.67(−2)	8.11(−3)	1.16(−2)	1.27(−2)	4.95(−3)	1.29(−3)	3.26(−4)	E_n
14	6.74(−2)	1.87(−2)	8.74(−3)	1.29(−2)	8.39(−3)	2.24(−3)	6.01(−4)	E_n
15	8.77(−2)	3.47(−2)	7.82(−3)	9.93(−3)	1.06(−2)	4.18(−3)	1.09(−4)	E_n
16	1.07(−1)	5.17(−2)	1.30(−2)	7.51(−3)	1.08(−2)	7.09(−3)	1.89(−2)	E_n

Table 1 ($\beta = 1/4$ Example 1, EMW)

k	n							
	8	16	32	64	128	256	512	
1	1.03(−3)	2.61(−4)	6.55(−5)	1.64(−5)	4.10(−6)	1.02(−6)	2.56(−7)	E_n
2	1.44(−3)	3.68(−4)	2.25(−5)	2.32(−5)	5.79(−6)	1.45(−6)	3.62(−7)	E_n
3	2.08(−4)	5.36(−4)	1.35(−4)	3.40(−5)	8.50(−6)	2.12(−6)	5.32(−7)	E_n
4	3.12(−3)	8.45(−4)	2.16(−4)	5.47(−5)	1.37(−6)	3.43(−6)	8.57(−7)	E_n
5	4.16(−3)	1.27(−3)	3.50(−4)	8.91(−5)	2.24(−5)	5.60(−6)	1.40(−6)	E_n
6	4.80(−3)	2.08(−3)	5.71(−4)	1.48(−4)	3.74(−5)	9.40(−6)	2.35(−6)	E_n
7	5.73(−3)	3.20(−3)	8.93(−4)	2.55(−4)	6.51(−5)	1.64(−6)	4.10(−6)	E_n
8	8.94(−3)	4.23(−3)	1.64(−3)	4.44(−4)	1.15(−4)	2.93(−5)	7.35(−6)	E_n
9	1.20(−2)	5.45(−3)	2.70(−3)	7.39(−4)	2.07(−4)	5.29(−5)	1.33(−5)	E_n
10	1.33(−2)	8.24(−3)	3.64(−3)	1.39(−4)	3.70(−4)	9.56(−5)	2.44(−5)	E_n
11	1.34(−2)	1.08(−2)	4.76(−3)	2.29(−3)	6.25(−4)	1.75(−5)	4.45(−5)	E_n
12	1.30(−2)	1.19(−2)	7.06(−3)	3.11(−3)	1.17(−3)	3.13(−4)	8.07(−5)	E_n
13	1.45(−2)	1.20(−2)	9.20(−3)	4.04(−3)	1.94(−3)	5.30(−4)	1.47(−4)	E_n
14	3.33(−2)	1.17(−2)	1.02(−2)	5.96(−3)	2.64(−3)	9.87(−4)	2.65(−4)	E_n
15	5.37(−2)	1.14(−2)	1.03(−2)	7.75(−3)	3.40(−3)	1.64(−3)	4.48(−4)	E_n

Table 2 (Example 1, $\beta = 1/4$, IEMW)

k	n							
	8	16	32	64	128	256	512	
1	1.31(−5)	8.45(−7)	5.32(−8)	3.33(−9)	2.08(−10)	1.31(−11)	9.79(−13)	E_n
2	2.63(−5)	1.73(−6)	1.10(−7)	6.90(−9)	4.31(−10)	2.71(−11)	3.62(−12)	E_n
3	6.71(−5)	4.80(−6)	3.13(−7)	1.97(−8)	1.23(−9)	7.73(−11)	4.76(−12)	E_n
4	2.91(−4)	2.58(−5)	1.75(−6)	1.12(−7)	7.08(−9)	4.43(−10)	2.77(−11)	E_n
5	8.50(−4)	1.09(−4)	9.24(−6)	6.40(−7)	4.11(−8)	2.58(−9)	1.62(−10)	E_n
6	1.09(−3)	3.05(−4)	3.85(−5)	3.08(−6)	2.07(−7)	1.32(−8)	8.28(−10)	E_n
7	3.46(−3)	7.16(−4)	1.14(−4)	1.15(−5)	8.65(−7)	5.73(−8)	3.63(−9)	E_n
8	8.29(−3)	1.36(−3)	2.62(−4)	3.54(−5)	3.18(−6)	2.30(−7)	1.50(−8)	E_n
9	1.37(−2)	3.15(−3)	6.02(−4)	9.72(−5)	1.04(−5)	8.56(−7)	6.00(−8)	E_n
10	1.88(−2)	6.72(−3)	1.12(−3)	2.19(−4)	3.02(−5)	2.87(−6)	2.25(−7)	E_n
11	2.15(−2)	1.12(−2)	2.57(−3)	4.99(−4)	8.15(−5)	8.90(−6)	7.73(−7)	E_n
12	1.62(−2)	1.59(−2)	5.77(−3)	9.17(−4)	1.83(−4)	2.53(−5)	2.477(−6)	E_n
13	1.49(−2)	1.90(−2)	9.34(−3)	2.134(−3)	4.10(−4)	6.75(−5)	7.45(−6)	E_n
14	1.52(−2)	1.54(−2)	1.32(−2)	4.91(−3)	7.64(−4)	1.51(−4)	2.09(−5)	E_n
15	3.59(−2)	1.22(−2)	1.62(−2)	7.83(−3)	1.79(−3)	3.38(−4)	5.58(−5)	E_n

Table 3 (Example 1, $\beta = 1/4$, New scheme)

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