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SET-THEORETIC RELATIONS AND BCH-ALGEBRAS WITH TRIVIAL STRUCTURE

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Abstract

In any BCH-algebra we can define a natural relation which is reflexive and anti-symmetric. This relation induces fundamental properties of a BCH-algebra, but not induces the BCH-operation in general. Moreover, some types of BCH-algebras may be obtained from other reflexive and anti-symmetric relations. We describe connections between such relations. We give also some methods of constructions of BCH-algebras from given relations.

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1. Introduction

In 1966 Y.Imai and K.Iséki [6], defined a class of algebras of (2,0) type, called *BCK-algebras* , which, on the one hand, generalizes the notion of the algebra of sets with the set subtraction as the fundamental non-nullary operation, and on the other hand the notion of the implication algebra [7]. BCK-algebras have many interesting generalizations such as BCI-algebras, BCC-algebras and BCH-algebras. Any such algebra has a certain natural order induced by its fundamental operation. Such order induces some properties of this operation, but this operation is not induced by this order in general. Moreover, such BCH-algebra may also be obtained from some other order. In this note we describe the connection between relations which create a BCH-algebra G and the natural order of G.

2. Orders and BCH-algebras

By an algebra $(G,\cdot,0)$ we mean a nonempty set G together with a binary multiplication (denoted by juxtaposition) and a certain distinguished element 0. Such algebra is called a *BCH-algebra* (or *CI-algebra* [1]) if the following conditions hold:

(1) $$xx = 0,$$

(2) $$(xy)z = (xz)y,$$

(3) $$xy = yz = 0 \text{ implies } x = y.$$

One can prove (cf. [3], [4], [5]) that every BCH-algebra satisfies

(4) $$x0 = x,$$

(5) $$0(xy) = (0x)(0y).$$

A BCH-algebra satisfying

(6) $$((xy)(xz))(zy) = 0$$

is called *a BCI-algebra.* A BCH-algebra is called *proper* (cf. [5]) if it is not a BCI-algebra, i.e. if it does not satisfy (6).

On any BCH-algebra $(G, \cdot, 0)$ one can define the so-called *natural order* by putting

$$(7) \qquad x \leq y \text{ iff } xy = 0.$$

This "order" is a reflexive and anti-symmetric relation, but, in general, it is not transitive.

Example 1. It is easily seen that $G = \{0, a, b, c\}$ with the multiplication defined by the table

	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	b	0

is a BCH-algebra. It is a proper because $((ac)(ab))(bc) \neq 0$. Its natural order is not transitive because $a \leq b$ and $b \leq c$ but $not(a \leq c)$.

If the natural order of a BCH-algebra $(G, \cdot, 0)$ has 0 as the smallest element, then $(G, \cdot, 0)$ is called *a BCH_0-algebra.* In other words, a BCH_0-algebra is a BCH-algebra $(G, \cdot, 0)$ in which

$$(8) \qquad 0x = 0$$

holds for all $x \in G$. A BCI-algebra satisfying (8) is called *a BCK-algebra.*

The natural order of a BCK-algebra $(G, \cdot, 0)$ is a partial order on G with 0 as a smallest element (cf. [7]). Moreover, any BCK-algebra $(G, \cdot, 0)$ may be considered (cf. [7]) as a groupoid $(G, \cdot, 0)$ with the natural order satisfying conditions: $0 \leq x$, $(xy)(xz) \leq zy$, $x0 = x$, $x \leq y \leq x$ imply $x = y$. Also any BCI-algebra is partially ordered by such natural order, but in this case 0 is not the smallest element in general.

On every set G equipped with a distinguished element 0 and a relation ρ we can define a binary multiplication in the following way

$$(9) \qquad x \cdot y = \begin{cases} 0 & \text{if } x\rho y \\ x & \text{otherwise} \end{cases}$$

We say that such algebra has *a trivial structure.* It is clear that any reflexive and anti-symmetric relation ρ yields a BCH_0-algebra. Any partial order on G with 0 as the smallest element defines on G the structure of a BCK-algebra.

Proposition 1. *If a BCH-algebra G has a trivial structure obtained from the reflexive and anti-symmmetric relation ρ, then its natural order coincides* *with ρ only in the case when ρ satisfies the minimum condition, i.e. if $0\rho x$ for every $x \in G$.*

Proof. If $x \leq y$ then $xy = 0$. This implies $x\rho y$, or $x = 0$. Since $0\rho y$ for all $y \in G$, then $x \leq y$ implies $x\rho y$, i.e. $\leq \subset \rho$. Conversely, if $x\rho y$ then by definition $xy = 0$, which gives $x \leq y$. Thus, $\rho \subset \leq$ and in the consequence $\rho = \leq$.□

Example 2. We will give an example where $\rho \neq \leq$. Let $G = \{0, a\}$ and let the reflexive and anti-symmetric relation ρ be given by $0\rho 0$, $a\rho a$, $not(0\rho a)$ and $not(a\rho 0)$. Then $(G, \cdot, 0)$ is a BCH_0-algebra with the trivial structure. Its multiplication is given by the following table:

$\cdot$	0	a
0	0	0
a	a	0

The natural order of $(G, \cdot, 0)$ satisfies $0 \leq a$. Hence $\rho \neq \leq$.

We say that a relation ρ defined on a set G with a distinguished element 0 is *locally reflexive* if $0\rho 0$, and *locally transitive* if $0\rho y$ and $y\rho z$ imply $0\rho z$.

Lemma 1. *Any relation satisfying the minimum condition is locally reflexive and locally transitive.*

Proposition 2. *If a relation satisfying the minimum condition induces on G the trivial structure of a BCH-algebra, then it is reflexive and anti-symmetric, and coincides with the natural order on this BCH-algebra.*

Proof. Assume that a relation ρ satisfies the minimum condition and defines on G a BCH-algebra $(G, \cdot, 0)$. If ρ is not reflexive, then there exists $x \in G$ such that $not(x\rho x)$. But in this case we have $x \cdot x = x$ by (9), and $x \cdot x = 0$, as $(G, \cdot, 0)$ is a BCH-algebra. Thus $x = 0$, which is in contradiction with local reflexivity.

If ρ is not anti-symmetric, then there exist $x, y \in G$, $x \neq y$ such that $x\rho y$ and $y\rho x$. Hence $x \cdot y = 0$ and $y \cdot x = 0$ by (9). But this by (3) implies $x = y$, which gives a contradiction. Thus, any relation satisfying the minimum condition and defining a BCH-algebra must be reflexive and anti-symmetric. By Proposition 1 such relation coincides with the natural order of this BCH-algebra. The proof is complete. □

Corollary 1. *If a relation ρ satisfies the minimum condition and induces on G the trivial structure of a BCK-algebra, then it is a partial order on G and coincides with the natural order on this BCK-algebra.*

The following example shows that a BCH-algebra may not be reproduced from its natural order.

Example 3. Consider three algebras defined on the set $G = \{0, a, b, c\}$ by the following tables:

$\circ$	0	a	b	c
0	0	0	0	0
a	a	0	c	c
b	b	0	0	b
c	c	0	0	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	0	0	b
c	c	0	0	0

$*$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	0	0	c
c	c	0	0	0

The first algebra is a proper BCH-algebra (cf. [5]). Its natural order is linear: $0 \leq c \leq b \leq a$. The BCH-algebra with the trivial structure defined on G by this order is given by the second table. The third algebra is a BCK-algebra obtained from this linear order by the construction given in [7]. It is not difficult to verify that these three algebras have the same natural order but are not isomorphic.

3. Constructions of BCH-algebras

Now we give some methods of constructions of BCH-algebras with the trivial structure from the given BCH-algebras (with the trivial structure). We start with a generalization of the construction obtained for BCK-algebras by H. Yutani [8].

Let $\{G_i\}_{i \in I}$ be a nonempty family of BCH-algebras such that $G_i \cap G_j = \{0\}$ for any distinct $i, j \in I$. In $\{G_i\}_{i \in I}$ we define a new multiplication identyfing it with a multiplication in any G_i, and putting $xy = x$ if belongs to distinct G_i. Direct computations show that the union $\bigcup_{i \in I} G_i$ is a BCH-algebra. It is called *the disjoint union of* $\{G_i\}_{i \in I}$ (cf. [3]).

In a general case where $\{G_i\}_{i \in I}$ is an arbitrary nonempty family of BCH-algebras, we consider $\{G_i \times \{i\}\}_{i \in I}$ and identify all $(0_i, i)$, where 0_i is a constant of G_i. By identifing each $x_i \in G_i$ with (x_i, i), the assumption of the definition mentioned above is satisfied. Consequently, we can define the disjoint union of an arbitrary BCH-algebra. Obviously, if all G_i have the trivial structure, then the disjoint union of $\{G_i\}_{i\in I}$ has also the trivial structure. Moreover, as a consequence of Theorem 5 from [3] we obtain

Proposition 3. *Let $\{S_i\}_{i\in I}$ be an indexed family of subsets of a BCH-algebra G with the trivial structure induced by the relation ρ. If*

$$(i)\quad G = \bigcup S_i\ ,$$
$$(ii)\quad S_i \cap S_j = \{0\}\ \text{ for any }\ i \neq j\ ,$$
$$(iii)\quad x \in S_i\ \text{ implies }\ \{y \in G : yx = 0\} \subset S_i\ \text{ for any }\ i \in I\ ,$$

then all S_i are subalgebras with the trivial structure induced by $\rho_i = \rho_{|S_i}$ and G is a disjoint union of S_i.

Also the following two constructions are a generalization of the known constructions for BCK-algebras. These constructions may be simply translated (by (9)) for BCH-algebras without the trivial structure.

Proposition 4. *Let $(G,\cdot,0)$ be a BCH-algebra with the trivial structure induced by ρ and let $a \notin G$. If we extend ρ to $G\cup\{a\}$ putting $a\rho a, 0\rho a, not(a\rho 0)$ and $a\rho x, not(x\rho a)$ for all $x \in G\backslash\{0\}$, then ρ induces on $G\cup\{a\}$ a BCH-algebra with the trivial structure. This new BCH-algebra is proper iff $(G,\cdot,0)$ is proper.*

Proposition 5. *Let $(G,\cdot,0)$ be a BCH-algebra with the trivial structure induced by ρ and let $a \notin G$. If we extend ρ to $G\cup\{a\}$ putting $a\rho a$, $x\rho a$ and $not(a\rho x)$ for all $x \in G$, then ρ induces on $G\cup\{a\}$ a BCH-algebra with the trivial structure. This BCH-algebra is proper iff $(G,\cdot,0)$ is proper.*

4. Ideals and congruences

A nonempty subset A of a BCI-algebra $(G,\cdot,0)$ is called *an ideal* iff $(i) 0 \in A$, $(ii) yx, x \in A$ *imply* $y \in A$. Obviously, any such ideal is a subalgebra of G and induces on G a congruence θ defined by $x\theta y$ iff $xy, yx \in A$. The set $G/\theta = \{C_x : x \in G\}$, where $C_x = \{y \in G : x\theta y\}$ with the operation $C_x * C_y = C_{xy}$ is a BCI-algebra. Unfortunately, this fact is not true for BCH-algebras.

Example 4. Let G be a proper BCH-algebra from Example 2 in [2]. Routine calculations prove that $A = \{0, b, d, f\}$ is an ideal of G, but the relation θ defined by this ideal is not a congruence because $c\theta e$ and $c\theta a$ not imply $cc\theta ea$. This gives a negative answer to the problem posed in [2]. On the other hand, one can prove that there exist congruences which are not defined by any ideal.

A special role in BCH-algebras play the congruences induced by some endomorphisms. It is not difficult to verify that the kernel of an endomorphism ϕ of a BCH-algebra $(G, \cdot, 0)$, i.e. the set $ker\phi = \{ x \in G : \phi(x) = 0\}$ is an ideal and the relation θ defined by $x\theta y$ iff $xy, yx \in ker\phi$, i.e. iff $\phi(x) = \phi(y)$ is a congruence. If ϕ has the form $\phi(x) = 0x$ (cf. (5)), then $G/ker\phi$ and $\phi(G)$ are isomorphic BCI-algebras (cf. [3]). These algebras are medial quasigroups. All such algebras with the finite set of generators are the direct product of the so-called cyclic BCI-algebras [3]. On the other hand, $\phi(G)$ is the largest (in the sense of inclusion) p-semisimple BCI-algebra contained in G. Similarly, $\{x \in G : \phi(x) = x\}$ is the largest Boolean group contained in G.

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