Univ. u Novom Sadu
Zb. Rad. Prirod.–Mat. Fak.
Ser. Mat. 25, 1 (1995), 83-87

Review of Research
Faculty of Science
Mathematics Series

SINGLE IDENTITY FOR A VARIETY OF 3-QUASIGROUPS

Zoran Stojaković
Institute of Mathematics, University of Novi Sad
Trg Dositeja Obradovića 4, 21000 Novi Sad, Yugoslavia

Abstract

We prove that the variety of 3-quasigroups equivalent to tetrahedral quadruple systems can be defined by a single identity.

AMS Mathematics Subject Classification (1991): 20N05, 05B07
Key words and phrases: tetrahedral quadruple system, alternating symmetric, 3-quasigroup, identity

1. Introduction

In [6] a class of quadruple systems called tetrahedral quadruple systems (TQSs) was defined. A TQS of order v is a pair (S,T), where S is a finite set of v elements and T is a family of directed quadruples $\langle abcd\rangle$, a,b,c,d distinct elements of S, such that every ordered triple of distinct elements of S belongs to exactly one directed quadruple from T. A directed quadruple $\langle abcd\rangle$ is the following set of 12 ordered triples

$$\begin{aligned}\langle abcd\rangle \quad &= \{\, (abc),(bca),(cab),(adb),(dba),(bad),\\ &\quad (acd),(cda),(dac),(bdc),(dcb),(cbd)\,\}\end{aligned}$$

It was proved in [6] that TQSs are equivalent to so-called generalized idempotent alternating symmetric (GIAS) 3-quasigroups, their properties were investigated and some parts of the spectrum of TQSs determined. In [4] further investigation of TQSs was carried on and it was proved that the spectrum of TQSs consists of all v such that $v \equiv 1, 2, 4, 5, 8, 10 \ (mod \ 12)$. TQSs represent a generalization of Steiner quadruple systems. In [7] Mendelsohn triple sytems derived from TQSs were considered.

A 3-groupoid (Q, f) is a GIAS-3-groupoid iff it satisfies the following identities

(1) $$f(x, y, f(x, z, y)) = z,$$

(2) $$f(f(x, y, z), y, x) = z,$$

(3) $$f(x, y, y) = x.$$

Hence the class of all GIAS-3-groupoids is a variety. Every GIAS-3-groupoid is necessarily a 3-quasigroup.

In [6] it is proved that finite GIAS-3-groupoids are equivalent to TQSs. If (S, T) is a TQS of order v, and f is defined for distinct elements $x, y, z \in S$ by

(4) $$f(x, y, z) = u \iff \langle xyzu \rangle \in T$$

and

$$f(x, y, y) = f(y, x, y) = f(y, y, x) = x,$$

otherwise, then (S, f) is GIAS-3-groupoid of order v. Conversely, if (S, f) is a GIAS-3-groupoid of order v, then by (4) a TQS (S, T) of order v is defined.

In [5] a single identity defining the algebra equivalent to Mendelsohn triple systems was given. Analogous results for algebras equivalent to Steiner triple systems and some of their subalgebras were obtained in [1], [2]. In [3] coordinatization of Steiner systems by algebras was discussed. In [8] it was proved that the varieties of algebras equivalent to Mendelsohn and Steiner quadruple systems can also be defined by single identities.

2. Single identity for GIAS-3-quasigroups

Here we shall show that GIAS-3-quasigroups can be defined by a single identity.

The following notation will be used. If (S, f) is a 3-groupoid, then the translation maps $T_1(a,b)$, $T_2(a,b)$, $T_3(a,b)$ can be defined by

$$T_1(y,z)(x) = T_2(x,z)(y) = T_3(x,y)(z) = f(x,y,z).$$

Theorem. *A 3-groupoid (S, f) is a GIAS iff the following identity is satisfied*

$$f(p,q,f(p,f(f(t,t,f(f(v,f(x,y,z),v),y,x)),u,u),q)) = z. \tag{5}$$

Proof. If (S, f) is a GIAS-3-groupoid, then it is easy to see that (5) holds.

Now, let (S, f) be a 3-groupoid such that (5) is valid. We shall prove that (1),(2) and (3) hold.

Identity (5) can be written by

$$T_3(p,q)T_2(p,q)T_1(u,u)T_3(t,t)T_1(y,x)T_2(v,v)T_3(x,y) = I, \tag{6}$$

where I is the identity mapping of S. From (6) it follows that $T_3(p,q)$ is onto and $T_3(x,y)$ is one-to-one, hence for all $x, y \in S$ $T_3(x,y)$ is a bijection. Hence

$$T_2(p,q)T_1(u,u)T_3(t,t)T_1(y,x)T_2(v,v) = T_3^{-1}(p,q)T_3^{-1}(x,y). \tag{7}$$

From (7) we get that $T_2(v,v)$ is a bijection. Putting in (7) $p = q$, we obtain

$$T_1(u,u)T_3(t,t)T_1(y,x) = T_2^{-1}(p,p)T_3^{-1}(p,p)T_3^{-1}(x,y)T_2^{-1}(v,v) \tag{8}$$

The preceding equality implies that $T_1(u,u)$ is a bijection.

From (6) it follows that for all $p, q, u, t, x, v, r, s \in S$

$$T_3(p,q)T_2(p,q)T_1(u,u)T_3(t,t)T_1(x,x)T_2(v,v)T_3(x,x) =$$

$$= T_3(r,s)T_2(r,s)T_1(u,u)T_3(t,t)T_1(x,x)T_2(v,v)T_3(x,x)$$

that is,

$$T_3(p,q)T_2(p,q) = T_3(r,s)T_2(r,s). \tag{9}$$

From (6) we get

$$T_3(p,p)T_2(p,p)T_1(u,u)T_3(t,t)T_1(x,x)T_2(v,v)T_3(x,x) =$$

$$= T_3(p,p)T_2(p,p)T_1(s,s)T_3(t,t)T_1(x,x)T_2(v,v)T_3(x,x)$$

which implies $T_1(u,u) = T_1(s,s)$. Analogously, we obtain that $T_2(v,v) = T_2(s,s)$ and $T_3(t,t) = T_3(s,s)$. Thus

$$f(z,x,x) = f(z,y,y), \;\; f(x,z,x) = f(y,z,y), \;\; f(x,x,z) = f(y,y,z),$$

and from the preceding equality for $y = z$ it follows

$$f(z,x,x) = f(x,z,x) = f(x,x,z) = f(z,z,z). \tag{10}$$

The equality (9) can be written by

$$f(p,q,f(p,z,q)) = f(r,s,f(r,z,s)). \tag{11}$$

Putting in the preceding equality $p = r = s = z,\;\; q = f(z,z,z)$, one gets

$$f(z,f(z,z,z),f(z,z,f(z,z,z))) = f(z,z,f(z,z,z)) = f(z,f(z,z,z),z),$$

and since $T_3(x,y)$ is a bijection, it follows

$$f(z,z,f(z,z,z)) = z. \tag{12}$$

From (11) for $r = s = z$, we get that

$$f(p,q,f(p,z,q)) = f(z,z,f(z,z,z)) = z,$$

which is identity (1). This identity can be written as $T_3(p,q)T_2(p,q) = I$ and since $T_3(p,q)$ is a bijection it follows that $T_2(p,q)$ is also a bijection and that $T_2(p,q)T_3(p,q) = I$.

Hence (6) reduces to

$$T_1(u,u)T_3(t,t)T_1(y,x)T_2(v,v)T_3(x,y) = I,$$

and putting in the preceding equality $x = y = v$ we get

$$T_1(u,u)T_3(t,t)T_1(v,v) = I$$

that is

$$f(f(t,t,f(z,v,v)),u,u) = z. \tag{13}$$

If in (13) we put $t = v = u = z$ it follows

$$f(f(z,z,f(z,z,z)),z,z) = z,$$

which by (12) gives $f(z,z,z) = z$. This and (10) imply

$$T_1(x,x) = T_2(x,x) = T_3(x,x) = I.$$

This means that (6) becomes

$$T_1(y,x)T_3(x,y) = I,$$

that is,

$$f(f(x,y,z),y,x) = z.$$

References

[1] Donovan, D., Single laws for two subvarieties of squags, Bull. Austral. Math. Soc. 42 (1990), no. 1, 157–165.

[2] Donovan, D., Oates-Williams, S., Single laws for sloops and squags, Discrete Math. 92 (1991), 79–83.

[3] Ganter, B., Werner, H., Co-ordinatizing Steiner systems, in "Topics on Steiner systems", Annals of Discrete Mathematics 7 (1980), 3–24.

[4] Hartman, A., Phelps, K. T., The spectrum of tetrahedral quadruple systems, Utilitas Math. 37 (1990), 181-188.

[5] Mendelsohn, N. S., A single groupoid identity for Steiner loops, Aequationes Math. 6 (1971), 228–230.

[6] Stojaković, Z., Madaras, R., On tetrahedral quadruple systems, Utilitas Math. 29 (1986),19–26.

[7] Stojaković, Z., Derived Mendelsohn triple systems, Bull. Austral. Math. Soc. 49 (1994), 413–419.

[8] Stojaković, Z., Single identities for Mendelsohn and Steiner 3-quasigroups (to appear).

Received by the editors December 13, 1994.