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STRONGLY DISTINGUISHED CONNECTIONS IN A RECURRENT K-HAMILTON SPACE

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Abstract

In the $n+mK$ dimensional differentiable manifold E^* (K-Hamilton space) special coordinate transformations are allowed. In $T^*(E^*) \otimes T^*(E^*)$ the metric tensor is given, and using the nonlinear connection $N, T(E^*)$ may be decomposed in $K+1$ orthogonal subspaces (with respect to G): $T_H(E^*)$ and ${}_{(\alpha)}T_V(E^*), \alpha = \overline{1,K}$. In $T(E^*)$ a strongly distinquished connection is introduced in such a way that Y and $\nabla_X Y$ belong to the same subspace of $T(E^*)$, $\forall X, Y \in T(E^*)$. The law of transformation of connection coefficients is given. For the metrical and recurrent case the connection coefficients and the torsion tensor are determined.

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1. Coordinate bases in $T(E^*)$ and $T^*(E^*)$

Let E^* be an $n+mK$ dimensional differentiable manifold. If u is one point of E^*, then in some local chart u has coordinates

$$u = ((x^i), (p_a^1), (p_a^2), \ldots, (p_a^K)) = ((x^i), (p_a^\alpha)) = (x, p),$$

where $(x^i) = (x^1, x^2, \ldots, x^n) = (x)$, $(p_a^\alpha) = ((p_1^\alpha), \ldots, (p_m^\alpha)) = (p^\alpha)$ and $a, b, c, d, e, f = \overline{1,m}, \quad i, j, h, k, l, m = \overline{1,n}, \quad \alpha, \beta, \gamma, \delta = \overline{1,K}$.

We shall consider the following transformation of a coordinate system. If $((x^{i'}),(p^{\alpha}_{a'})) = (x', p')$ are the coordinates of the same point u in the new coordinate system, then

$$(1.1)\quad \begin{aligned} &\text{(a)}\ x^{i'} = x^{i'}(x^1,\dots x^n) && \operatorname{rank}[\partial x^{i'}/\partial x^i] = n \\ &\text{(b)}\ p^{\alpha}_{a'} = \overset{(\alpha)}{M}{}^{a}_{a'}(x^1,\dots,x^n)p^{\alpha}_{a} && \operatorname{rank}[\partial p^{\alpha}_{a'}/\partial p^{\alpha}_{a}] = m. \end{aligned}$$

The Einstein summation convention will be used for all three kinds of indices, except when the index is in brackets. If (1.1) is valid, then an inverse transformation exists

$$(1.2)\quad \text{(a)}\ x^{i} = x^{i}(x^{1'},\dots x^{n'}) \qquad \text{(b)}\ p^{\alpha}_{a} = \overset{(\alpha)}{M}{}^{a'}_{a}(x^{1'},\dots,x^{n'})p^{\alpha}_{a'}.$$

The natural basis $\bar{B} = \{(\partial_i),(\partial^a_1),\dots,(\partial^a_K)\}$ of $T(E^*)$ is formed by n vectors of type $\partial_i = \partial/\partial x^i$ and $m \cdot K$ vectors of type $\partial^a_\alpha = \partial/\partial p^\alpha_a$. Any vector field $X \in T(E^*)$ may be written in the form

$$(1.3)\quad X = \bar{X}^i \partial_i + \bar{X}^\alpha_a \partial^a_\alpha.$$

With respect to the coordinate transformations (1.1) and (1.2) the basic vectors of $\overline{B}$ have the following law of transformation

$$(1.4)\quad \begin{bmatrix} \partial_i \\ \partial^a_1 \\ \vdots \\ \partial^a_K \end{bmatrix} = \begin{bmatrix} \frac{\partial x^{i'}}{\partial x^i} & (\partial_i \overset{(1)}{M}{}^a_{a'})p^1_a & \dots & (\partial_i \overset{(K)}{M}{}^a_{a'})p^K_a \\ 0 & \overset{(1)}{M}{}^a_{a'} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \overset{(K)}{M}{}^a_{a'} \end{bmatrix} \begin{bmatrix} \partial_{i'} \\ \partial^{a'}_1 \\ \vdots \\ \partial^{a'}_K \end{bmatrix}$$

$$(1.5)\quad \begin{bmatrix} \partial_{i'} \\ \partial^{a'}_1 \\ \vdots \\ \partial^{a'}_K \end{bmatrix} = \begin{bmatrix} \frac{\partial x^{j}}{\partial x^{i'}} & (\partial_{i'} \overset{(1)}{M}{}^{b'}_{b})p^1_{b'} & \dots & (\partial_{i'} \overset{(K)}{M}{}^{b'}_{b})p^K_{b'} \\ 0 & \overset{(1)}{M}{}^{a'}_{b} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \overset{(K)}{M}{}^{a'}_{b} \end{bmatrix} \begin{bmatrix} \partial_{j} \\ \partial^{b}_1 \\ \vdots \\ \partial^{b}_K \end{bmatrix}.$$

Substituting (1.5) into (1.4), we obtain

$$(1.6)\qquad \begin{aligned} &(a)\ \frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial x^{j}}{\partial x^{i'}} = \delta_i^j \qquad (b)\ \overset{(\alpha)}{M}{}^a_{a'}\overset{(\alpha)}{M}{}^{a'}_b = \delta^a_b \\ &(c)\ (\partial_{i'}\overset{(\alpha)}{M}{}^{b'}_b)p^\alpha_{b'}\frac{\partial x^{i'}}{\partial x^i} + (\partial_i \overset{(\alpha)}{M}{}^a_{a'})p^\alpha_a \overset{(\alpha)}{M}{}^{a'}_b = 0. \end{aligned}$$

(1.6c) is the consequence of (1.4) and (1.6b).

2. Adapted bases in $T(E^*)$ and $T^*(E^*)$

From (1.4) and (1.5) it is obvious that ∂_i and $\partial_{i'}$ are not transformed as tensors, so we introduce a new, the so-called adapted basis $B = \{(\delta_i), (\partial^a_1), \dots, (\partial^a_K)\}$ of $T(E^*)$, where by definition

$$(2.1)\qquad \delta_i = \partial_i - N^\alpha_{ai}(x,p)\partial^a_\alpha$$

and $N^\alpha_{ai}(x,p)$ are the coefficients of the nonlinear connection. Under coordinate transformation (1.1) and (1.2), they transform in the following way:

$$(2.2)\qquad \begin{aligned} &(a)\ N^\alpha_{a'i'}(x',p') = \overset{(\alpha)}{M}{}^a_{a'}\frac{\partial x^i}{\partial x^{i'}}N^\alpha_{ai}(x,p) - p^\alpha_a \frac{\partial \overset{(\alpha)}{M}{}^a_{a'}}{\partial x^i}\frac{\partial x^i}{\partial x^{i'}} \\ &(b)\ N^\alpha_{bj}(x,p) = \overset{(\alpha)}{M}{}^{b'}_b \frac{\partial x^{j'}}{\partial x^j}N^\alpha_{b'j'}(x',p') + p^\alpha_a \overset{(\alpha)}{M}{}^{a'}_b \frac{\partial \overset{(\alpha)}{M}{}^a_{a'}}{\partial x^j}. \end{aligned}$$

Any vector field $X \in T(E^*)$ in the adapted basis B is given by

$$(2.3)\qquad X = X^i\delta_i + X^\alpha_a\partial^a_\alpha.$$

The coordinates of the vector X given by (2.3) and the elements of basis B are transformed as tensor in the following way:

$$(2.4)\qquad \begin{aligned} &(a)\ \delta_i = \frac{\partial x^{i'}}{\partial x^i}\delta_{i'} \qquad (b)\ \partial^a_\alpha = \overset{(\alpha)}{M}{}^a_{a'}(x)\partial^{a'}_\alpha \\ &(c)\ X^i = \frac{\partial x^i}{\partial x^{i'}}X^{i'} \quad (d)\ X^\alpha_a = \overset{(\alpha)}{M}{}^{a'}_a(x')X^\alpha_{a'}. \end{aligned}$$

From (1.3) and (2.3) we obtain the relation between the coordinates of the field X in the bases $\bar{B}$ and B. They are connected by the formula

$$X^i = \bar{X}^i, \quad X^\alpha_a = \bar{X}^\alpha_a + N^\alpha_{ai}\bar{X}^i$$

The subspace of $T(E^*)$ spanned by $\{\delta_i\}$ shall be denoted by $T_H(E^*)$ (horizontal part) and the subspace spanned by $\{\partial^a_\alpha\}$ by ${}_{(\alpha)}T_V(E^*)$ (the vertical α part). So we have

$$T(E^*) = T_H(E^*) \oplus T_V(E^*),$$

where

$$T_V(E^*) = \sum_{\alpha=1}^{K} {}_{(\alpha)}T_V(E^*),$$

$$\dim T_H(E^*) = n, \quad \dim_{(\alpha)} T_V(E^*) = m.$$

$X^i\delta_i$ is the horizontal and $X^\alpha_a\partial^a_\alpha$ the vertical part of the field X.

Now (2.3) may be written in the form

$$X = X_H + X_V, \quad X_H = X^i\delta_i, \quad X_V = X^\alpha_a\partial^a_\alpha.$$

Let us consider the dual tangent space of E^*, the space $T^*(E^*)$. The natural basis in $T^*(E^*)$ is

$$\begin{aligned}\overline{B}^* &= \{dx^1, \dots, dx^n, dp^1_1, \dots, dp^1_m, \dots, dp^K_1, \dots dp^K_m\} \\ &= \{dx^i, dp^1_a, \dots, dp^K_a\}.\end{aligned}$$

From (1.1) we obtain

(2.5) (a) $$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i$$

(b) $$dp^\alpha_{a'} = \frac{\partial \overset{(\alpha)}{M}{}^a_{a'}(x)}{\partial x^i} p^\alpha_a dx^i + \overset{(\alpha)}{M}{}^a_{a'}(x) dp^\alpha_a.$$

From (2.5b) it is obvious that dp^α_a are not transformed as tensors, so we introduce a new basis $B^* = \{(dx^1), (\delta p^1_a), \dots, (\delta p^K_a)\}$, where

(2.6) $$\delta p^\alpha_a = dp^\alpha_a + N^\alpha_{ai}(x,p)dx^i.$$

By the coordinate transformation (1.1) the bases $\overline{B}^*$ and B^* are related by (2.5a) and

(2.7) (a) $\delta p^\alpha_a = \overset{(\alpha)}{M}{}^{a'}_{a}(x')\delta p^\alpha_{a'}$ (b) $\delta p^\alpha_{a'} = \overset{(\alpha)}{M}{}^{a}_{a'}(x)\delta p^\alpha_a.$

The proof of (2.7) is obtained using (2.6) and (2.2). Any field $w \in T^*(E^*)$ can be written in the bases and B^* and $\overline{B}^*$ in the following way

$$w = \bar{w}_i dx^i + \bar{w}^a_\alpha dp^\alpha_a = w_i dx^i + w^a_\alpha \delta p^\alpha_a, \tag{2.8}$$

where

$$w_i = \bar{w}_i - N^\alpha_{a_i} \bar{w}^a_\alpha, \qquad \bar{w}^a_\alpha = w^a_\alpha. \tag{2.9}$$

The subspace of $T^*(E^*)$ spanned by $\{(dx^i)\}$ shall be denoted by $T^*_H(E^*)$ and the subspace spanned by $\{(\delta p^\alpha_a)\}$ by ${}_\alpha T^*_V(E^*)$.

So we have

$$T^*(E^*) = T^*_H(E^*) \oplus T^*_V(E^*),$$

where

$$T^*_V(E^*) = \sum_{\alpha=1}^{K} {}_\alpha T^*_V(E^*).$$

Now (2.8) may be written in the form

$$w = w_H + w_V, \;\; w_H = w_i dx^i, \;\; w_V = w^a_\alpha \delta p^\alpha_a.$$

If $\{(dx^i), (\delta p^1_a), ..., (\delta p^K_{a'})\}$ and $\{(dx^{i'}), (\delta p^1_{a'}), ..., (\delta p^K_{a'})\}$ are two bases in $T^*(E^*)$ related by (2.5a) and (2.7) then any $w \in T^*(E^*)$ satisfies the relation

$$w = w_i dx^i + w^a_\alpha \delta p^\alpha_a = w_{i'} dx^{i'} + w^{a'}_\alpha \delta p^\alpha_{a'}. \tag{2.10}$$

Substitutingg $dx^{i'}$ from (2.5a) and $\delta p^\alpha_{a'}$ from (2.7b) into (2.10) and comm-paring the coefficients besides basis vectors, we obtain

$$w_i = w_{i'} \frac{\partial x^{i'}}{\partial x^i}, \;\; w^a_\alpha = \overset{(\alpha)}{M}{}^a_{a'} \, w^{a'}_\alpha. \tag{2.11}$$

By a straightforward calculation we can prove

Proposition 2.1. *The adapted bases* $\{(\delta_i), (\partial^a_1), ..., (\partial^a_K)\}$ *and* $\{(dx^i), (\delta p^1_a), ..., (\delta p^K_a)\}$ *are dual to each other, i.e.*

$$\begin{aligned} &< \delta_i, dx^j >= \delta^j_i \quad < \delta_i, \delta p^\alpha_a >= 0 \\ &< \partial^a_\alpha, dx^j >= 0 \quad < \partial^a_\alpha, \delta p^\beta_b >= \delta^a_b \delta^\beta_\alpha. \end{aligned}$$

3. Tensor Fields on E^*

a) A horizontal tensor fiels t_H has the local representation:

$$t_H = t^{i_1 \dots i_p}_{\quad j_1 \dots j_q}(x,p)\delta_{i_1} \otimes \dots \otimes \delta_{i_p} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}.$$

It is defined on

$$\underbrace{T_H(E^*) \otimes \dots \otimes T_H(E^*)}_{p \ times} \otimes \underbrace{T^*_H(E^*) \otimes \dots \otimes T^*_H(E^*)}_{q \ times}.$$

By changing the coordinates given by (1.1) and (1.2), the coordinates of the field t_H have the following transformation law

$$t^{i'_1 \dots i'_p}_{\quad j'_1 \dots j'_q} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \dots \frac{\partial x^{i'_p} \partial x^{j_1}}{\partial x^{i_p} \partial x^{j'_1}} \dots \frac{\partial x^{j_q}}{\partial x^{j'_q}}, \quad t^{i_1 \dots i_p}_{\quad j_1 \dots j_q}.$$

b) The α verical tensor field ${}_{(\alpha)}t_V$ has the local representation

$${}_{(\alpha)}t_V = {}_{(\alpha)}\, t^{\alpha \dots \alpha a_1 \dots a_r}_{b_1 \dots b_s \alpha \dots \alpha} \partial^{b_1}_{\alpha} \otimes \dots \partial^{b_s}_{\alpha} \otimes \delta p^{\alpha}_{a_1} \otimes \dots \otimes \delta p^{\alpha}_{a_r}$$

(not summing over α).

${}_{(\alpha)}t_V$ is defined on

$$\underbrace{{}_{(\alpha)}T_V(E^*) \otimes \dots \otimes_{(\alpha)} T_V(E^*)}_{s \ times} \otimes \underbrace{{}_{(\alpha)}T^*_V(E^*) \otimes \dots \otimes_{(\alpha)} T^*_V(E^*)}_{r \ times}.$$

By changing the coordinates of type (1.1) and (1.2) the coordinates of the field ${}_{(\alpha)}t_V$ given above have the following transformation law

$${}_{(\alpha)}t^{a'_1 \dots a'_r \alpha \dots \alpha}_{\alpha \dots \alpha b'_1 \dots b'_s} =$$

$$= {}_{(\alpha)}\, t^{a_1 \dots a_r \alpha \dots \alpha}_{\alpha \dots \alpha b_1 \dots b_s} \overset{(\alpha)}{M}{}^{a'_1}_{a_1} \dots \overset{(\alpha)}{M}{}^{a'_r}_{a_r} \overset{(\alpha)}{M}{}^{b_1}_{b'_1} \dots \overset{(\alpha)}{M}{}^{b_s}_{b'_s}.$$

c) A vertical tensor field t_V on $T_V(E^*) \otimes T^*_V(E^*)$ has the form

$$t_V = t^{\alpha b_1}_{a_1 \beta} \partial^{a_1}_{\alpha} \otimes \delta p^{\beta}_{b_1}$$

(summing over α and β).

The coordinate transformation of tensor t_V is given by

$$t^{\alpha\ b'_1}_{a'_1\ \beta} = t^{\alpha\ b_1}_{a_1\ \beta}\ \overset{(\alpha)}{M}{}^{a_1}_{a'_1}\ \overset{(\beta)}{M}{}^{b'_1}_{b_1}\ .$$

d) A tensor field t on

$$\underbrace{T_H(E^*)\otimes ...\otimes T_H(E^*)}_{p\ times}\otimes\underbrace{T^*_H(E^*)\otimes ...\otimes T^*_H(E^*)}_{q\ times}\otimes$$

$$\underbrace{T_V(E^*)\otimes ...\otimes T_V(E^*)}_{s\ times}\otimes\underbrace{T^*_V(E^*)\otimes ...\otimes T^*_V(E^*)}_{r\ times}$$

is given by

$$t = t^{i_1...i_p\qquad\ \beta_1...\beta_s a_1...a_r}_{\qquad j_1...j_q\ b_1...b_s\alpha_1...\alpha_r}(x,p)\delta_{i_1}\otimes ...\otimes\delta_{i_p}\otimes$$

$$dx^{j_1}\otimes ...\otimes dx^{j_q}\otimes\partial^{b_1}_{\beta_1}\otimes ...\otimes\partial^{b_s}_{\beta_s}\otimes\delta p^{\alpha_1}_{a_1}\otimes ...\otimes\delta p^{\alpha_r}_{a_r}.$$

The summation goes over all the indices.

The coordinate transformation of the above tensor is given by

$$t = t^{i'_1...i'_p\qquad\ \beta_1...\beta_s a'_1...a'_r}_{\qquad j'_1...j'_q\ b'_1...b'_s\alpha_1...\alpha_r} = t^{i_1...i_p\qquad\ \beta_1...\beta_s a_1...a_r}_{\qquad j_1...j_q\ b_1...b_s\alpha_1...\alpha_r}$$

$$\frac{\partial x^{i'_1}}{\partial x^{i_1}}...\frac{\partial x^{i'_p}\partial x^{j_1}}{\partial x^{i_p}\partial x^{j'_1}}...\frac{\partial x^{j_q}}{\partial x^{j'_q}}\ \overset{(\beta_1)}{M}{}^{b_1}_{b'_1}\ ...\ \overset{(\beta_s)}{M}{}^{b_s}_{b'_s}\overset{(\alpha_1)}{M}{}^{a_1}_{a_1}\ ...\ \overset{(\alpha_r)}{M}{}^{a'_r}_{a_r}$$

The order of spaces $T_H(E^*)$, $T^*_H(E)$, ${}_{(\alpha)}T_V(E^*)$ and $T^*_V(E^*)$ can be taken arbitrary. It has an influence on the order of indices of tensor t which is defined on their tensor product.

4. Metric tensor in the K-Hamilton space

In the space $T^*(E^*)\otimes T^*(E^*)$ the metric tensor G with respect to the basis $\{(dx^i),(\delta p^1_a,...,(\delta p^K_a)\}$ has the form

(4.1) $$G = [(dx^i)(\delta p^1_a)...(\delta p^K_a)].$$

$$\begin{bmatrix} [g_{ij}] & [g_{i1}^{\ b}] & \cdots & [g_{iK}^{\ b}] \\ [g^a_{1j}] & [g^{ab}_{11}] & \cdots & [g^{ab}_{1K}] \\ \vdots & \vdots & \vdots & \vdots \\ [g^a_{Kj}] & [g^{ab}_{K1}] & \cdots & [g^{ab}_{KK}] \end{bmatrix} \otimes \begin{bmatrix} dx^j \\ \delta p^1_b \\ \vdots \\ \delta p^K_b \end{bmatrix}.$$

The matrices $[g_{ij}]$, $[g_{i\beta}^{\ b}]$, $[g^a_{\alpha j}]$ and $[g^{ab}_{\alpha\beta}]$ have the format $n \times n$, $n \times m$, $m \times n$ and $m \times m$ respectively. As G is a tensor, its coordinates in the new coordinate system (x', p') are transformed in the following way:

(4.2)
$$a)\quad g_{i'j'} = g_{ij}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} \qquad b)\ g^{a'}_{\alpha j'} = g^a_{\alpha j}\overset{(\alpha)}{M^{a'}_a}\frac{\partial x^j}{\partial x^{j'}}$$

$$c)\quad g_{i'\beta}^{\ b'} = g_{i\beta}^{\ b}\frac{\partial x^i}{\partial x^{i'}}\overset{(\beta)}{M^{b'}_b} \qquad d)\quad g^{a'b'}_{\alpha\beta} = g^{ab}_{\alpha\beta}\overset{(\alpha)}{M^{a'}_a}\overset{(\beta)}{M^{b'}_b}$$

We shall suppose that G is a symmetric, positive definite tensor field of rank $n + mK$. From the symmetry it follows that

$$g_{ji} = g_{ij} \qquad g_{i\beta}^{\ b} = g^b_{\beta i} \qquad g^{ab}_{\alpha\beta} = g^{ba}_{\beta\alpha}.$$

The "covariant" coordinates of the field $X = X^i\delta_i + X^\alpha_a\partial^a_\alpha$ are given by

(4.3) $$X_i = g_{ij}X^j + g_{i\alpha}^{\ a}X^a_\alpha, \quad X^a_\alpha = g^a_{\alpha i}X^i + g^{ab}_{\alpha\beta}X^\beta_b.$$

The inverse matrix of G (appearing in (4.1)) is given by

$$\begin{bmatrix} [g^{jk}] & [g^{j1}_{\ \ c}] & \cdots & [g^{jK}_{\ \ c}] \\ [g^{1k}_b] & [g^{11}_{bc}] & \cdots & [g^{1K}_{bc}] \\ \vdots & \vdots & \vdots & \vdots \\ [g^{Kk}_b] & [g^{K1}_{bc}] & \cdots & [g^{KK}_{bc}] \end{bmatrix}$$

The matrices $[g^{jk}]$, $[g^{j\gamma}_{\ \ c}]$, $[g^{\gamma k}_c]$ and $[g^{\beta\gamma}_{bc}]$ have the format $n \times n$, $n \times m$, $m \times m$ and $m \times m$ respectively. Now we have

(4.4)
$$a)\quad g_{ij}g^{jk} + g_{i\beta}^{\ b}g^{\beta k}_b = \delta^k_i \quad b)\quad a^a_{\alpha j}g^{j\gamma}_{\ \ c} + g^{ab}_{\alpha\beta}g^{\beta\gamma}_{bc} = \delta^a_c\delta^\gamma_\alpha$$

$$c)\quad g_{ij}g^{j\gamma}_{\ \ c} + g_{i\beta}^{\ b}g^{\beta\gamma}_{bc} = 0 \quad d)\quad g^a_{\alpha j}g^{jk} + g^{ab}_{\alpha\beta}g^{\beta k}_b = 0.$$

The contravariant coordinates of $w = w_i dx^i + w^a_\alpha \delta p^\alpha_a$ are given by

$$w^i = g^{ij} w_j + g^{i\alpha}{}_a w^a_\alpha \qquad w^\alpha_a = g^{j\alpha}{}_a w_j + g^{\alpha\beta}_{ab} w^b_\beta. \tag{4.5}$$

Using (2.4), (2.11) and (4.2) it can be shown that following transformation laws are valid:

$$X_{i'} = X_i \frac{\partial x^i}{\partial x^{i'}} \quad X^{a'}_\alpha = X^a_\alpha \overset{(\alpha)}{M^{a'}_a}$$

$$w_{i'} = w_i \frac{\partial x^i}{\partial x^{i'}} \quad w^\alpha_{a'} = w^\alpha_a \overset{(\alpha)}{M^a_{a'}} .$$

If the $k-$Hamilton function $H(X,p)$ is given in the space E^*, then the metric tensor G can be defined in the following way:

$$g_{ij}(x,p) = g_{ij}(x) \qquad g_{i\beta}{}^b = 0 \qquad g^a_{\alpha j} = 0$$

$$g^{ab}_{\alpha\beta} = 2^{-1} \partial^a_\alpha \partial^b_\beta H^2(x,p) \quad \forall\, \alpha, \beta = \overline{1,K},$$

where $g_{ij}(x)$ is some metric tensor defined on M and M is the π^* projection of E^*

$$\pi^*(E^*) = M, \quad \pi^*((x^i),(p^1_a),...,(p^K_a)) = (x^i).$$

We can not define

$$g_{i\beta}{}^b(x,p) = 2^{-1}\delta_i \partial^b_\beta H^2(x,p), \quad g_{ij}(x,p) = 2^{-1}\delta_i\delta_j H^2(x,p),$$

because the above quantities are not transformed as tensor.

Using the metric G determined by (4.1) we define the scalar product (X,Y) of fields $X, Y \in T(E^*)$ by

$$(X,Y) = g_{ij} X^i Y^j + g_{i\beta}{}^b X^i Y^\beta_b + g^a_{\alpha j} X^\alpha_a Y^j + g^{ab}_{\alpha\beta} X^\alpha_a Y^\beta_b. \tag{4.6}$$

Then lenght of $X, |X|$ is defined by $|X|^2 = (X,X)$ and $cos\theta$, where θ is the angle between X and Y by

$$cos\ \theta = \frac{(X,Y)}{|X|\,|Y|}. \tag{4.7}$$

When $cos\ \theta = 0$, we say that the fields X and Y are orthogonal to each other. For the horizontal field X_H we have

$$X_H = X^i \delta_i, \quad |X_H|^2 = g_{ij} X^i X^j$$

and for vertical vector X_V we have

$$X_V = X^\alpha_a \partial^a_\alpha \quad |X_V|^2 = g^{ab}_{\alpha\beta} X^\alpha_a X^\beta_b .$$

For the field ${}_{(\alpha)}X_V \in_{(\alpha)} T_V(E^*)$ we have

$${}_{(\alpha)}X_V = X^\alpha_a \partial^a_\alpha, \quad |{}_{(\alpha)}X_V|^2 = g^{ab}_{\alpha\alpha} X^\alpha_a X^\alpha_b (\text{not summing over } \alpha).$$

Theorem 4.1. *The necessary and sufficient conditions that the subspaces $T_H(E^*)$, ${}_{(1)}T_V(E^*), ...,_{(K)} T_V(E^*)$ of $T(E^*)$ should be orthogonal to each other with respect to the metric tensor G, are*

$$[g_{i\beta}{}^b] = 0, \quad [g^a_{\alpha j}] = 0, \quad \forall\, \alpha, \beta = \overline{1, K} \; \alpha \neq \beta.$$

Definition 4.1. *The differentiable manifold E^* in which the coordinate transformations of type (1.1) and (1.2) are allowed, supplied with the non-linear connection N (see (2.2)) and the metric tensor G (given by (4.1)) is called the $K-$ Hamilton space.*

5. Strongly distinguished connection in $T(E^*)$

The distinguished connection ∇ or $d-$ connection in the $K-$ Hamilton space in [15], [9], [16] and others is defined as a function $\nabla : (X, Y) \to \nabla_X Y$; X, Y, $\nabla_X Y \in T(E^*)$ for which, besides the usual conditions for the linear connection, the following restrictions hold

$$(5.1) \quad (a) \qquad \nabla_X Y_H \in T_H(E^*) \quad (b) \nabla_X Y_V \in T_V(E^*)$$

$$\forall\, X \in T(E^*), \forall\, Y_H \in T_H(E^*) \quad \text{and} \quad \forall\, Y_V \in T_V(E^*).$$

The strongly distinguished or s.d. - connection is the linear connection for which (5.1 a) and (5.2)

$$(5.2) \qquad \nabla_X \, {}_{(\alpha)}Y_V \in \; {}_{(\alpha)} T_V(E^*)$$

hold.

Definition 5.1. *The strongly distingushed connection ∇ in $T(E^*)$ is the linear connection defined by*

$$(5.3) \qquad (a) \quad \nabla_{\delta_i}\delta_j = F_{j\,i}^{k}\delta_k, \qquad b)\ \nabla_{\delta_i}\partial_\alpha^a = \overset{(\alpha)}{F^{a}}{}_{ci}\,\partial_\alpha^c$$

$$(c) \quad \nabla_{\partial_\alpha^a}\delta_j = C_j{}^{ka}_{\ \alpha}\delta_k \quad d)\ \nabla_{\partial_\alpha^a}\partial_\beta^b = \overset{(\beta)}{C}{}^{b}{}_{c\alpha}{}^{a}\,\partial_\beta^c.$$

In (5.3b) and (5.3d) there is no summation over α and β respectively.

Proposition 5.1. *If $X, Y \in T(E^*)$, where X is given by (2.3) and $Y = Y^j\delta_j + Y_b^\beta\partial_\beta^b$, then*

$$(5.4) \qquad \nabla_X Y = (Y^j_{|i}X^i + Y^j|^a_\alpha X^\alpha_a)\delta_j + (Y^\beta_{b|i}X^i + Y_b^\beta|^a_\alpha X^\alpha_a)\partial_\beta^b,$$

where

$$(a) \quad Y^j_{|i} = \delta_i Y^j + F^j_{k\,i}Y^k$$

$$(b) \quad Y^j|^a_\alpha = \partial^a_\alpha Y^j + C^{\ ja}_{k\ \alpha}Y^k$$

$$(5.5) \qquad (c) \quad Y^\beta_{b|i} = \delta_i Y_b^\beta + \overset{(\beta)}{F^{d}}{}_{bi}\,Y_d^\beta$$

$$(d) \quad Y_b^\beta|^a_\alpha = \partial^a_\alpha Y_b^\beta + \overset{(\beta)}{C^{d}}{}_{b\alpha}{}^{a}\,Y_d^\beta.$$

Proposition 5.2. *If $((x^i), (p^\alpha_a))$ and $((x^{i'}), (p^\alpha_{a'}))$ are two coordinate systems connected by (1.1) and (1.2) then*

$$(5.6) \qquad \nabla_{X'}Y' = \nabla_X Y$$

iff $Y^j_{|i}$, $Y^j|^a_\alpha$, $Y^\beta_{b|i}$ and $Y_b^\beta|^a_\alpha$ are transformed as tensors, i.e.

$$Y^{j'}_{|i'} = Y^j_{|i}\,(\partial_{i'}x^i)(\partial_j x^{j'}) \qquad Y^{j'}|^{a'}_\alpha = Y^j|^a_\alpha(\partial_j x^{j'})\,\overset{(\alpha)}{M}{}^{a'}_a$$

$$Y^\beta_{b'|i'} = Y^\beta_{b|i}\,\overset{(\beta)}{M}{}^b_{b'}\,(\partial_{i'}x^i) \qquad Y^\beta_{b'}|^{a'}_\alpha = Y_b^\beta|^a_\alpha\,\overset{(\beta)}{M}{}^b_{b'}\overset{(\alpha)}{M}{}^{a'}_a$$

or equivalently iff the s.d.-connection coefficients have the following law of transformation:

(a) $F^k_{ji} = F^{k'}_{j'i'}(\partial_j x^{j'})(\partial_{k'} x^k)(\partial_i x^{i'}) + (\partial_i \partial_j x^{k'})(\partial_{k'} x^k)$

(b) $\overset{(\beta)}{F}{}^b{}_{ci} = \overset{(\beta)}{F}{}^{b'}{}_{c'i'} \overset{(\beta)}{M}{}^b_{b'} \overset{(\beta)}{M}{}^{c'}_c (\partial_i x^{i'}) + (\partial_i \overset{(\beta)}{M}{}^b_{b'}) \overset{(\beta)}{M}{}^{d'}_c$

(5.7)

(c) $C^{k\alpha}_{j\ a} = C^{k'\ \alpha'}_{j'\ a'}(\partial_j x^{j'})(\partial_{k'} x^k) \overset{(\alpha)}{M}{}^a_{a'}$

(d) $\overset{(\beta)}{C}{}^{b\ a}_{\ c\alpha} = \overset{(\beta)}{C}{}^{b'\ a'}_{\ c'\alpha} = \overset{(\beta)}{M}{}^b_{b'} \overset{(\beta)}{M}{}^{c'}_c \overset{(\alpha)}{M}{}^a_{a'}$.

Proof. The proof is obtained by direct calculation using (5.4) - (5.6) , (2.1) and (2.4).

It follows from (5.7) that the $F's$ which appear in (5.7a) and (5.7 b) are transformed as connection coefficients, and the two $C's$ in (5.7c) and (5.7b) as tensors.

The torsion tensor $T(X,Y)$ for the s.d.-connection, is as usual, given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]. \quad \square \tag{5.8}$$

Theorem 5.1. *In the k−Hamilton space the torsion tensor for the s.d. - connection has the form*

$$\begin{aligned} T(X,Y) = \; & (F^j_{ki} - F^j_{ik}) X^i Y^k \delta_j + C^{j\ a}_{k\ \alpha} X^\alpha_a Y^k \delta_j - C^{jb}_{i\beta} X^i Y^\beta_b \delta_j + \\ & \overset{(\beta)}{C}{}^{d\ a}_{\ b\alpha} X^\alpha_a Y^\beta_d \partial^b_\beta - \overset{(\alpha)}{C}{}^{a\ b}_{\ c\beta} X^\alpha_a Y^\beta_b \partial^c_\alpha + \\ & (\partial_i N^\alpha_{aj} - \partial_j N^\alpha_{ai} + N^\beta_{bj} \partial^b_\beta N^\alpha_{ai} - N^\beta_{bi} \partial^b_\beta N^\alpha_{aj}) X^i Y^j \partial^a_\alpha + \\ & \left[\overset{(\beta)}{F}{}^d{}_{bi} Y^\beta_b - (\partial^a_\alpha N^\beta_{bi}) Y^\alpha_a \right] X^i \partial^b_\beta + \\ & \left[(\partial^a_\alpha N^\beta_{bj}) X^\alpha_a - \overset{(\beta)}{F}{}^d{}_{bj} X^\beta_d \right] Y^j \partial^b_\beta . \end{aligned} \tag{5.9}$$

Proof. The proof is obtained by direct calculation using (5.3), (5.4), (5.5) and (5.8). □

If we suppose that the nonlinear connection N is such that N^β_{bi} is the function only of (x) and $(p^\beta) = (p^\beta_1, ..., p^\beta_m)$ for $\forall \beta = \overline{1,K}$, then the three last lines in (5.9) have the form

$$(\partial_i N^{\alpha}_{aj} - \partial_j N^{\alpha}_{ai} + N^{(\alpha)}_{bj}\partial^b_{(\alpha)} N^{\alpha}_{bi} - N_b{}^{(\alpha)}_i \partial^b_{(\alpha)} N^{\alpha}_{aj}) X^i Y^j \partial^a_{\alpha} +$$

(5.10)
$$+(\overset{(\beta)}{F}{}^d{}_{bi} - \partial^d_{(\beta)} N^{(\beta)}_{bi}) X^i Y^{\beta}_d \partial^b_{\beta} +$$

$$+(\partial^d_{(\beta)} N^{(\beta)}_{bj} - \overset{(\beta)}{F}{}^d{}_{bj}) X^{\beta}_d Y^j \partial^b_{\beta}.$$

The Einstein summation convection is not meant on (α) and (β). □

Theorem 5.2. *In the $K-$ Hamilton space the torsion tensor for the s.d. - connection is identically equal to zero iff*

(a) $\partial^a_{\alpha} N^b_{\beta i} = 0$ *for* $\forall \alpha \neq \beta$

(b) $F^j_{ki} - F^j_{ik} = 0,$

(c) $C_k{}^j{}^a_{\alpha} = 0$

(5.11)

(d) $\overset{(\alpha)}{C}{}^d{}^a_{b\alpha} - \overset{(\alpha)}{C}{}^a{}^d_{b\alpha} = 0$

(e) $\overset{(\alpha)}{C}{}^a{}^c_{b\gamma} = 0$ *for* $\forall \alpha \neq \gamma$

(f) $\partial^i N^{\alpha}_{aj} - \partial_j N^{\alpha}_{ai} - N^{(\alpha)}_{bi} \partial^b_{(\alpha)} N^{\alpha}_{aj} + N^{(\alpha)}_{bj} \partial^b_{(\alpha)} N^{\alpha}_{ai} = 0$

(g) $\overset{(\beta)}{F}{}^d{}_{bi} = \partial^d_{(\beta)} N^{\beta}_{bi}.$

Proof. The proof follows from (5.9) and (5.10). □

Theorem 5.3. *In the $K-$Hamilton space the torsion free s.d. - connection has the following properties*

(a) $\nabla_{\delta_i}\delta_j = \nabla_{\delta_j}\delta_i, \ \nabla_{X_H} Y_H \in T_H(E^*)$

(b) $\nabla_{(\alpha)X_V}\ {}_{(\beta)}Y_V = 0 \ \forall \alpha \neq \beta$

(5.12)

(c) $\nabla_{\partial^a_\alpha}\partial^b_\alpha = \nabla_{\partial^b_\alpha}\partial^a_\alpha \qquad \nabla_{(\alpha)X_V\,(\alpha)}Y_V \in \quad {}_{(\alpha)}T_V(E^*)$

(d) $\nabla_{(\alpha)\,X_V}Y_H = 0.$

If instead of (5.11 a) the stronger condition

$$\partial^a_\alpha N^b_{\beta i} = 0 \qquad \forall \alpha \neq \beta = \overline{1,m} \tag{5.13}$$

is satisfied, then besides (5.12) the torsion free s.d. - connection has the property $\nabla_{X_H}(\alpha)Y_V = 0$.

From Theorems 5.2 and 5.3 follows the following

Theorem 5.4. *In the torsion free K– Hamilton space for which condition (5.13) holds the strongly distinguished connection reduces to the following form:*

$$\nabla_{\delta_i}\delta_j = F^k_{ji}\delta_k,\ \ F^k_{ij} = F^k_{ji},\ \ \nabla_{\delta_i}\partial^a_\alpha = 0,\ \ \nabla_{\partial^a_\alpha}\delta_j = 0,$$

$$\nabla_{\partial^a_\alpha}\partial^b_\beta = 0 \ \ \text{for } \alpha \neq \beta,\ \ \nabla_{\partial^a_\alpha}\partial^b_\alpha = \overset{(\alpha)}{C}{}^{b\ \ a}_{\ c}\ \partial^c_\alpha,\ \ \overset{(\alpha)}{C}{}^{b\ \ a}_{\ c} = \overset{(\alpha)}{C}{}^{a\ \ b}_{\ c}, \tag{5.14}$$

6. Strongly distinguished connection in $T^*(E^*)$

The connection ∇ defined on $T(E^*)$ by (5.3) induces a connection ∇^* on $T^*(E^*)$ which will also be denoted by ∇. For the field X defined by (2.9) and w defined by (2.14) we have

$$\nabla_X w = \nabla_{X^i\delta_i + X^\alpha_a\partial^a_\alpha}(w_j dx^j + w^b_\beta \delta p^\beta_b). \tag{6.1}$$

Definition 6.1. *The connection ∇ on $T^*(E^*)$ is defined by*

$$\text{(a)}\ \ \nabla_{\delta_i}dx^j = \tilde{F}_k{}^j{}_i dx^k \ \ \text{(b)}\ \ \nabla_{\delta_i}\delta p^\beta_b = \overset{(\beta)}{\tilde{F}}{}^c{}_{bi}\ \delta p^\beta_c$$

(6.2)

$$\text{(c)}\ \ \nabla_{\partial^a_\alpha}dx^j = \tilde{C}_k{}^j{}^a_\alpha dx^k \ \ \text{(d)}\nabla_{\partial^a_\alpha}\delta p^\beta_b = \overset{(\beta)}{\tilde{C}}{}^c{}_{b\alpha}^{\ a}\ \delta p^\beta_c.$$

Proposition 6.1. *If $X \in T(E^*)$ and $w \in T^*(E^*)$, then*

$$\nabla_X w = (w_{j|i}X^i + w_j|^a_\alpha X^\alpha_a)dx^j + (w^b_{\beta|i}X^i + w^b_\beta|^a_\alpha w^\alpha_a)\delta p^\beta_b, \tag{6.3}$$

where

$$(a)\quad w_{j|i} = \delta_i w_j + \tilde{F}_j{}^k{}_i w_k$$

$$(b)\quad w_j|^a_\alpha = \partial^a_\alpha w_j + \tilde{C}_j{}^{ka}_\alpha w_k$$

$$(6.4)$$

$$(c)\quad w^b_{\beta|i} = \delta_i w^b_\beta + \overset{(\beta)}{\tilde{F}^b}{}_{ci}\, w^c_\beta$$

$$(d)\quad w^b_\beta|^a_\alpha = \partial^a_\alpha w^b_\beta + \overset{(\beta)}{\tilde{C}^b}{}_{c\alpha}^{\ a}\, w^c_\beta.$$

Proof. Substituting (6.2) into (6.1) and using the linarity of connection ∇ we obtain (6.3) and (6.4). □

Using the properties of the linear connection ∇ and relations (4.1) and (6.2), the following relations are fulfilled:

$$
(6.5)\quad
\begin{aligned}
\nabla_X G \;=\; & (g_{ij|k}X^k + g_{ij}\,|^c_\gamma\, X^\gamma_c)dx^i \otimes dx^j + \\
& (g^{\ b}_{i\beta|k}X^k + g^{\ b}_{i\beta}\,|^c_\gamma\, X^\gamma_c)dx^i \otimes \delta p^\beta_b + \\
& (g^{a}_{\alpha j|k}X^k + g^{a}_{\alpha j}\,|^c_\gamma\, X^\gamma_c)\delta p^\alpha_a \otimes dx^j + \\
& (g^{ab}_{\alpha\beta|k}X^k + g^{ab}_{\alpha\beta}\,|^c_\gamma\, X^\gamma_c)\delta p^\alpha_a \otimes \delta p^\beta_b,
\end{aligned}
$$

where

$$(\text{a})\quad g_{ij|k} = \delta_k g_{ij} + g_{hj}\tilde{F}_i{}^h{}_k + g_{ih}\tilde{F}_j{}^h{}_k$$

$$(\text{b})\quad g_{ij}|^c_\gamma = \partial^c_\gamma g_{ij} + g_{hj}\tilde{C}_i{}^{hc}_{\ \gamma} + g_{ih}\tilde{C}_j{}^{hc}_{\ \gamma}$$

$$(\text{c})\quad g^{\ b}_{i\beta|k} = \delta_k g^{\ b}_{i\beta} + g^{\ b}_{h\beta}\tilde{F}^h_{ik} + g^{\ d}_{i\beta}\,\overset{(\beta)}{\tilde{F}^b}{}_{dk}$$

$$(6.6)$$

$$(\text{d})\quad g^{\ b}_{i\beta}|^c_\gamma = \partial^c_\gamma g^{\ b}_{i\beta} + g^{\ b}_{h\beta}\tilde{C}^{hc}_{i\gamma} + g^{\ d}_{i\beta}\,\overset{(\beta)^{b\,c}}{\tilde{C}}{}_{d\gamma}$$

$$(\text{e})\quad g^{ab}_{\alpha\beta|k} = \delta_k g^{ab}_{\alpha\beta} + g^{db}_{\alpha\beta}\,\overset{(\alpha)}{\tilde{F}^a}{}_{dk} + g^{ad}_{\alpha\beta}\,\overset{(\alpha)}{\tilde{F}^b}{}_{dk}$$

$$(\text{f})\quad g^{ab}_{\alpha\beta}|^c_\gamma = \partial^c_\gamma g^{ab}_{\alpha\beta} + g^{db}_{\alpha\beta}\,\overset{(\alpha)}{\tilde{C}^a}{}_{d\gamma}^{\ c} + g^{ab}_{\alpha\beta}\,\overset{(\beta)}{\tilde{C}^b}{}_{d\gamma}^{\ c}.$$

Definition 6.2. *The $K-$Hamilton space will be called recurrent if there exists a field $\lambda(x,p)=\lambda_k(x,p)dx^k+\lambda^c_\gamma(x,p)\delta p^\gamma_c$, such that*

$$
\begin{aligned}
&g_{ij|k}=\lambda_k g_{ij} && g_{ij}|^c_\gamma=\lambda^c_\gamma g_{ij}\\
&g^{\ b}_{i\beta|k}=\lambda_k g^{\ b}_{i\beta} && g^{\ b}_{i\beta}|^c_\gamma=\lambda^c_\gamma g^{\ b}_{i\beta}\\
&g^{a}_{\alpha j|k}=\lambda_k g^{a}_{\alpha j} && g^{a}_{\alpha j}|^c_\gamma=\lambda^c_\gamma g^{a}_{\alpha j}\\
&g^{ab}_{\alpha\beta|k}\ \lambda_k g^{ab}_{\alpha\beta} && g^{ab}_{\alpha\beta}|^c_\gamma=\lambda^c_\gamma g^{ab}_{\alpha\beta}.
\end{aligned}
\tag{6.7}
$$

The $K-$Hamilton space will be called a metric space if

$$
\begin{aligned}
&g_{ij|k}=0 \qquad g_{ij}|^c_\gamma=0 \qquad g^{\ b}_{i\beta|k}=0 \qquad g^{\ b}_{i\beta}|^c_\gamma=0\\
&g^{a}_{\alpha j|k}=0 \qquad g^{a}_{\alpha j}|^c_\gamma=0 \qquad g^{ab}_{\alpha\beta|k}=0 \qquad g^{ab}_{\alpha\beta}|^c_\gamma=0.
\end{aligned}
\tag{6.8}
$$

Theorem 6.1. *In the recurrent and in the metric $K-$Hamilton spaces the strongly distinguished connection ∇ defined on $T(E^*)$ by (5.3) and on $T^*(E^*)$ by (6.2) is compatible with the raising and lowering of the indices by the metric tensor G iff following conditions are satisfied*

$$
\begin{aligned}
&\tilde{F}_{k\ i}^{\ j}=-F_{k\ i}^{\ j} \quad \overset{(\beta)}{\tilde{F}}{}^{c}_{\ bi}=-\overset{(\beta)}{F}{}^{c}_{\ bi},\\
&\tilde{C}_{k\ \alpha}^{\ ja}=-C_{k\ \alpha}^{\ ja}, \quad \overset{(\beta)}{\tilde{C}}{}^{c\ a}_{\ b\alpha}=-\overset{(\beta)}{C}{}^{c\ a}_{\ b\alpha}.
\end{aligned}
\tag{6.9}
$$

Proof. The proof is obtained from (4.3), (6.6), (6.7), (6.4) and (5.5). It is similar to the proof of Theorem 6.1 in [8]. □

Theorem 6.2. *In the recurrent $K-$Hamilton space the coordinates of the inverse metric tensor satisfy the following relations*

$$
\begin{aligned}
&g^{ij}_{|k}=-\lambda_k g^{ij}, \quad g^{i\beta}_{\ \ b|k}=-\lambda_k g^{i\beta}_{\ \ b}, \quad g^{\delta\beta}_{db|k}=-\lambda_k g^{\delta\beta}_{db}\\
&g^{ij}|^c_\gamma=-\lambda^c_\gamma g^{ij}, \quad g^{i\beta}_{\ \ b}|^c_\gamma=-\lambda^c_\gamma g^{i\beta}_{\ \ b}, \quad g^{\delta\beta}_{db}|^c_\gamma=-\lambda^c_\gamma g^{\delta\beta}_{db}.
\end{aligned}
\tag{6.10}
$$

Proof. The proof follows from (4.4) and is the same as the proof of Theorem 6.2 in [8.] □

To avoid confusion in the application of the Einstein summation convention in the case when more than two indices are equal, we introduce the following notation:

$$F_{\alpha dk}^{a\delta} = \begin{cases} \overset{(\alpha)}{F^{a}}{}_{dk} & \text{for} \quad \alpha = \delta \\ 0 & \text{for} \quad \alpha \neq \delta \end{cases}$$

$$C_{\alpha d\gamma}^{a\delta c} = \begin{cases} \overset{(\alpha)}{C^{a}}{}_{d\gamma}^{c} & \text{for} \quad \alpha = \delta \\ 0 & \text{for} \quad \alpha \neq \delta \end{cases}$$

Using the above notation the rasing and lowering of the middle index of the connection coefficients are given by the following formulae:

(6.11)

$$\begin{bmatrix} F_{ijk} \\ F_{i\alpha k}^{a} \end{bmatrix} = \begin{bmatrix} g_{ij} & g_{j\delta}^{d} \\ g_{\alpha h}^{a} & g_{\alpha\delta}^{ad} \end{bmatrix} \begin{bmatrix} F_{ik}^{h} \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} F_{ik}^{h} \\ 0 \end{bmatrix} = \begin{bmatrix} g^{hj} & g_{a}^{h\alpha} \\ g_{d}^{\delta j} & g_{d\alpha}^{\delta\alpha} \end{bmatrix} \begin{bmatrix} F_{ijk} \\ F_{i\alpha k}^{\ a} \end{bmatrix}$$

$$\begin{bmatrix} F_{\alpha ik}^{a} \\ F_{\alpha\beta k}^{ab} \end{bmatrix} = \begin{bmatrix} g_{ih} & g_{i\delta}^{c} \\ g_{\delta h}^{b} & g_{\beta\gamma}^{bc} \end{bmatrix} \begin{bmatrix} 0 \\ F_{\alpha ck}^{a\gamma} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 \\ F_{\alpha dk}^{a\delta} \end{bmatrix} = \begin{bmatrix} g^{hi} & g^{h\beta}_{\ b} \\ g_{d}^{\delta i} & g_{db}^{\delta\beta} \end{bmatrix} \begin{bmatrix} F_{\alpha ik}^{a} \\ F_{\alpha\beta k}^{ab} \end{bmatrix}$$

$$\begin{bmatrix} C_{ij\gamma}^{\ \ c} \\ C_{i\alpha\gamma}^{\ ac} \end{bmatrix} = \begin{bmatrix} g_{jl} & g_{j\delta}^{d} \\ g_{\alpha l}^{a} & g_{\alpha\delta}^{ad} \end{bmatrix} \begin{bmatrix} C_{i\ \gamma}^{\ lc} \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} C_{i\ \gamma}^{\ hc} \\ 0 \end{bmatrix} = \begin{bmatrix} g^{hj} & g^{h\alpha}_{\ a} \\ g_{d}^{\delta j} & g_{d\alpha}^{\delta\alpha} \end{bmatrix} \begin{bmatrix} C_{ij\gamma}^{\ \ c} \\ C_{i\alpha\gamma}^{\ ac} \end{bmatrix}$$

$$\begin{bmatrix} C_{\alpha i\gamma}^{a\ c} \\ C_{\alpha\beta\gamma}^{abc} \end{bmatrix} = \begin{bmatrix} g_{ih} & g_{i\epsilon}^{e} \\ g_{\beta h}^{b} & g_{\beta\epsilon}^{be} \end{bmatrix} \begin{bmatrix} 0 \\ C_{\alpha e\gamma}^{a\epsilon c} \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 \\ C_{\alpha d\gamma}^{a\delta c} \end{bmatrix} = \begin{bmatrix} g^{hi} & g^{h\beta}_{\ b} \\ g_{d}^{\delta i} & g_{bd}^{\delta\beta} \end{bmatrix} \begin{bmatrix} C_{\alpha i\gamma}^{a\ c} \\ C_{\alpha\beta\gamma}^{abc} \end{bmatrix}$$

The above equations are well defined because they are compatible with (4.4)

Using (6.11), (6.6) may be written in the following form:

(a) $g_{ij|k} = \delta_k g_{ij} - F_{ijk} - F_{jik}$

(b) $g_{ij}|_{\gamma}^{c} = \partial_{\gamma}^{c} g_{ij} - C_{ij\gamma}^{\ \ c} - C_{ji\gamma}^{\ \ c}$

(6.12)

(c) $g_{i\beta|k}^{\ b} = \delta_k g_{i\beta}^{\ b} - F_{i\beta k}^{\ b} - F_{\beta ik}^{b}$

(d) $g_{i\beta}^{\ b}|_{\gamma}^{c} = \partial_{\gamma}^{c} g_{i\beta}^{\ b} - C_{i\beta\gamma}^{\ bc} - C_{\beta i\gamma}^{b\ c}$

(e) $g^{ab}_{\alpha\beta|k} = \delta_k g^{ab}_{\alpha\beta} - F^{ab}_{\alpha\beta k} - F^{ba}_{\beta\alpha k}$

(f) $g^{ab}_{\alpha\beta}|^c_\gamma = \partial^c_\gamma g^{ab}_{\alpha\beta} - C^{abc}_{\alpha\beta\gamma} - C^{bac}_{\beta\alpha\gamma}$.

7. Strongly distingushed connection coefficients in the recurrent $K-$ Hamilton space

Theorem 7.1. *In the recurrent $K-$Hamilton space supplied with the metric tensor G, arbitrary torsion tensor T, the strongly distinguished connection coefficients are determined by (7.1) - (7.8):*

(7.1) $$2F_{ijk} = (\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki}) - (\lambda_k g_{ij} + \lambda_i g_{jk} - \lambda_j g_{ki}) + A$$

where

$$\begin{aligned} A &= (F_{ijk} - F_{kji}) + (F_{kij} - F_{jik}) + (F_{ikj} - F_{jki}) = \\ &= g_j{}^h(F^h_{ik} - F^h_{ki}) + g_{ih}(F^h_{kj} - F^h_{jk}) + g_{kh}(F^h_{ij} - F^h_{ji}). \end{aligned}$$

(7.2) $$\begin{aligned} 2F^a_{i\alpha k} &= (\delta_k g^{\ a}_{i\alpha} + \delta_i g^a_{\alpha k} - \partial^a_\alpha g_{ik}) - \\ &-(\lambda_k g^a_{\alpha i} + \lambda_i g^a_{\alpha k} - \lambda^a_\alpha g_{ik}) + B, \end{aligned}$$

$$\begin{aligned} B &= (F^{\ a}_{i\alpha k} - F^{\ a}_{k\alpha i}) - (F^a_{\alpha ki} - C_{ik\alpha}^{\ \ \ a})) - (F^a_{\alpha ik} - C_{ki\alpha}^{\ \ \ a}) + \\ &= g^a_{\alpha h}(F^h_{ik} - F^h_{ki}) - (g^{\ c}_{k\gamma} F^{a\gamma}_{\alpha ci} - g_{kh} C_{i\ \alpha}^{\ ha}) - \\ &- (g^{\ c}_{i\gamma} F^{a\gamma}_{\alpha ck} - g_{ik} C_{k\ \alpha}^{\ ha}). \end{aligned}$$

(7.3) $$\begin{aligned} 2F^a_{\alpha ik} &= (\delta_k g^a_{\alpha i} + \delta^a_\alpha g_{ik} - \delta_i g^{\ a}_{k\alpha}) - \\ &-(\lambda_k g^a_{\alpha i} + \lambda^a_\alpha g_{ik} - \lambda_i g^{\ a}_{k\alpha}) - B. \end{aligned}$$

(7.4) $$\begin{aligned} 2F^{ab}_{\alpha\beta k} &= (\delta_k g^{ab}_{\alpha\beta} + \partial^a_\alpha g^b_{\beta k} - \partial^b_\beta g^{\ a}_{k\alpha}) - \\ &-(\lambda_k g^{ab}_{\alpha\beta} + \lambda^a_\alpha g^b_{\beta k} - \lambda^b_\beta g^{\ a}_{k\alpha} + C, \end{aligned}$$

where

$$C = (F^{ab}_{\alpha\beta k} - C_{k\beta a}^{\ ba}) - (F^{ba}_{\beta\alpha k} - C_{k\alpha\beta}^{\ ab}) + (C^{a\ b}_{\alpha k\beta} - C^{b\ a}_{\beta k\alpha}) =$$

$$= (g^{bc}_{\beta\gamma} F^{a\gamma}_{\alpha ck} - g^{b}_{\beta k} C^{\ ha}_{k\ \alpha}) - (g^{ac}_{\alpha\gamma} F^{b}_{\beta ck} - g^{a}_{\alpha h} C^{\ hb}_{k\ \beta}) + g^{\ e}_{k\epsilon}(C^{a\epsilon b}_{\alpha e\beta} - C^{b\epsilon a}_{\beta e\alpha})$$

(7.5)
$$2C_{ij\alpha}^{\ \ a} = (\partial^a_\alpha g_{ij} + \delta_i g_{j\alpha}^{\ a} - \delta_j g^a_{\alpha i}) -$$
$$-(\lambda^a_\alpha g_{ij} + \lambda_i g_{j\alpha}^{\ a} - \lambda_j g^a_{\alpha i}) + D,$$

where

$$D = (F_{i\alpha j}^{\ a} - F_{j\alpha i}^{\ a}) + (F^a_{\alpha ij} - C_{ji\alpha}^{\ \ a}) - (F^a_{\alpha ji} - C_{ij\alpha}^{\ \ a}) -$$
$$- g^a_{\alpha h}(F^h_{ij} - F^h_{ji}) + (g_{i\gamma}^{\ c} F^{a\gamma}_{\alpha cj} - g_{ih} C_{j\ \alpha}^{\ ha}) -$$
$$- (g_{j\gamma}^{\ c} F^{a\gamma}_{\alpha ci} - g_{jh} C_{i\ \alpha}^{\ ha}).$$

(7.6)
$$2C_{k\alpha\beta}^{\ ab} = (\partial^b_\beta g_{k\alpha}^{\ a} + \delta_k g^{ab}_{\alpha\beta} - \partial^a_\alpha g^b_{\beta k}) -$$
$$-(\lambda^b_\beta g_{k\alpha}^{\ a} + \lambda_k g^{ab}_{\alpha\beta} - \partial^a_\alpha g^b_{\beta k}) - E,$$

where

$$E = (F^{ab}_{\alpha\beta k} - C_{k\beta\alpha}^{\ ba}) + (F^{ba}_{\beta\alpha k} - C_{k\alpha\beta}^{\ ab}) + (C^{a\ b}_{\alpha k\beta} - C^{b\ a}_{\beta k\alpha}) =$$
$$= (g^{bc}_{\beta\gamma} F^{a\gamma}_{\alpha ck} - g^{b}_{\beta h} C^{\ ha}_{k\ \alpha}) + (g^{ac}_{\alpha\gamma} F^{b\gamma}_{\beta ck} - g^{a}_{\alpha h} C^{\ hb}_{k\ \beta}) + g^{\ e}_{k\epsilon}(C^{a\epsilon b}_{\alpha\epsilon\beta} - C^{b\epsilon a}_{\beta\epsilon\alpha})$$

(7.7)
$$2C^{a\ b}_{\alpha k\beta} = (\partial^b_\beta g^{\ a}_{\alpha k} + \partial^a_\alpha g_{k\beta}^{\ b} - \delta_k g^{ba}_{\beta\alpha} -$$
$$-(\lambda^b_\beta g^{\ a}_{\alpha k} + \lambda^a_\alpha g_{k\beta}^{\ b} - \lambda_k g^{ba}_{\beta\alpha}) + E$$

(7.8)
$$2C^{abc}_{\alpha\beta\gamma} = (\partial^c_\gamma g^{ab}_{\alpha\beta} + \partial^a_\alpha g^{bc}_{\beta\gamma} - \partial^b_\beta g^{ca}_{\gamma\alpha}) -$$
$$-(\lambda^c_\gamma g^{ab}_{\alpha\beta} + \lambda^a_\alpha g^{bc}_{\beta\gamma} - \lambda^b_\beta g^{ca}_{\gamma\alpha}) + F,$$

where

$$F = (C^{abc}_{\alpha\beta\gamma} - C^{cba}_{\gamma\beta\alpha}) + (C^{acb}_{\alpha\gamma\beta} - C^{bca}_{\beta\gamma\alpha}) + (C^{cab}_{\gamma\alpha\beta} - C^{bac}_{\beta\alpha\gamma}).$$

Proof. The proof follows from (6.7), (6.11) and (6.12). □

In (7.1) - (7.9) the expressions A, B, C, D, E, F and G are functions of torsion tensor T and the nonlinear connection N. The connection coefficients in the K recurent Hamilton space can be determined using an arbitrary temsor. In the usual terminology, that the space is torsion free when $T(X,Y) = 0\ \forall X, Y \in T(E^*)$ is used, then we have

Theorem 7.2. *In the torsin free recurrent $K-$Hamilton space, supplied with the metric tensor G, the strongly distinguished connection coefficients are determined by (7.1) - (7.8) where A, B, C, D, E and F have the following value:*

$$(7.9) \qquad A = 0,$$

$$B = -(g_{k(\alpha)}^{\ c}\partial_\alpha^a N_{ci}^{(\alpha)} + g_{i(\alpha)}^{\ c}\partial_\alpha^a N_{ck}^{(\alpha)},$$

$$C = g_{\beta k(\alpha)}^{bc}\partial_\alpha^a N_{ck}^{(\alpha)} - g_{\alpha(\beta)}^{ac}\partial_\beta^b N_{ck}^{(\beta)},$$

$$D = g_{i(\alpha)}^{\ c}\partial_\alpha^a N_{cj}^{(\alpha)} - g_{j(\alpha)}^{\ c}\partial_\alpha^a N_{ci}^{(\alpha)},$$

$$E = g_{\beta(\alpha)}^{bc}\partial_\alpha^a N_{ck}^{(\alpha)} - g_{\alpha(\beta)}^{ac}\partial_\beta^b N_{ck}^{(\beta)}$$

and $\delta_i N_{aj}^{\alpha} - \delta_j N_{ai}^{\alpha} = 0$ *(which follows from (5.11 f)).*

Proof. The proof follows from Theorem 7.1 and Theorem 7.2. □

If in (7.1) - (7.8) the filed $\lambda = \lambda_k dx^k + \lambda_\alpha^a \delta p_a^\alpha$ is equal to zero, then the recurrent $K-$ Hamilton space becomes a $K-$ Hamilton space supplied with a strongly distinguished metric connection.

Theorem 7.3. *In the torsion free $K-$Hamilton space supplied with the metric tensor G, the strongly distinguished metric connection coefficients are given by*

$$(a) \quad 2F_{ijk} = (\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki})$$

$$(b) \quad 2F_{i\alpha k}^{\ a} = (\delta_k g_{i\alpha}^{\ a} + \delta_i g_{\alpha k}^{a} - \partial_\alpha^a g_{ik}) - (g_{k(\alpha)}^{\ c}\partial_\alpha^a N_{ci}^{(\alpha)} + g_{i(\alpha)}^{\ c})\partial_\alpha^a N_{ck}^{(\alpha)})$$

$$(c) \quad 2F_{\alpha ik}^{a} = (\delta_k g_{\alpha i}^{a} + \partial_\alpha^a g_{ik} - \delta_i g_{k\alpha}^{\ a}) + (g_{k(\alpha)}^{\ c}\partial_\alpha^a N_{ci}^{(\alpha)} + g_{i(\alpha)}^{\ c}\partial_\alpha^a N_{cj}^{(\alpha)})$$

$$(d) \quad 2F_{\alpha\beta k}^{ab} = (\delta_k g_{\alpha\beta}^{ab} + \partial_\alpha^a g_{\beta k}^{b} - \partial_\beta^b g_{k\alpha}^{\ a}) + (g_{\beta(\alpha)}^{b\,c}\partial_\alpha^a N_{ck}^{(\alpha)} - g_{\alpha(\beta)}^{a\,c}\partial_\beta^b N_{ck}^{(\beta)})$$

$$(7.10)$$

$$(e) \quad 2C_{ij\alpha}^{\ \ a} = (\partial_\alpha^a g_{ij} + \delta_i g_{j\alpha}^{\ a} - \delta_j g_{\alpha i}^{a}) + (g_{i(\alpha)}^{\ c}\partial_\alpha^a N_{cj}^{(\alpha)} - g_{j(\alpha)}^{\ c}\partial_\alpha^a N_{ci}^{(\alpha)})$$

$$(f) \quad 2C_{k\alpha\beta}^{\ ab} = (\partial_\beta^b g_{k\alpha}^{\ a} + \delta_k g_{\alpha\beta}^{ab} - \partial_\alpha^a g_{\beta k}^{b}) - (g_{\beta(\alpha)}^{b\,c}\partial_\alpha^a N_{ck}^{(\alpha)} + g_{\alpha(\beta)}^{a\,c}\partial_\alpha^a N_{ck}^{(\beta)})$$

$$(g) \quad 2C_{\alpha k\beta}^{a\ b} = (\partial_\beta^b g_{\alpha k}^{a} + \partial_\alpha^a g_{k\beta}^{\ b} - \delta_k g_{\beta\alpha}^{ba}) + (g_{\beta(\alpha)}^{b\,c}\partial_\alpha^a N_{ck}^{(\alpha)} + g_{\alpha(\beta)}^{a\,c}\partial_\beta^b N_{ck}^{(\beta)})$$

$$(h) \quad 2C_{\alpha\beta\gamma}^{abc} = (\partial_\gamma^c g_{\alpha\beta}^{ab} + \partial_\alpha^a g_{\beta\gamma}^{bc} - \partial_\beta^b g_{\gamma\alpha}^{ca})$$

$$\delta_i N^{\alpha}_{aj} - \delta_j N^{\alpha}_{ai} = 0.$$

Proof. The proof follows from Theorem 7.1 and Theorem 7.2. □

Theorem 7.4. *In the torsion free metric $K-$Hamilton space in which $T_H(E^*)$ is orthogonal to $T_V(E^*)$, i.e. where the metric tensor G has the property $[g_{k\alpha}^{\ a}] = 0 \ \forall \ \alpha = \overline{1,K}$ the strongly distinguished connection coefficients have the form*

(a) $2F_{ijk} = (\delta_k g_{ij} + \delta_i g_{jk} - \delta_j g_{ki})$

(b) $2F_{i\alpha k}^{\ a} = -\partial^a_\alpha g_{ik}$

(c) $2F^a_{\alpha ik} = \partial^a_\alpha g_{ik}$

(d) $2F^{ab}_{\alpha\beta k} = \delta_k g^{ab}_{\alpha\beta} + g^{b\ c}_{\beta(\alpha)}\partial^a_\alpha N^{(\alpha)}_{ck} - g^{a\ c}_{\alpha(\beta)}\partial^b_\beta N^{(\beta)}_{ck}$

(7.11)

(e) $2C_{ij\alpha}^{\ \ a} = \partial^a_\alpha g_{ij}$

(f) $2C_{k\alpha\beta}^{\ ab} = \delta_k g^{ab}_{\alpha\beta} - (g^{b\ c}_{\beta(\alpha)}\partial^a_\alpha N^{(\alpha)}_{ck} + g^{a\ c}_{\alpha(\beta)}\partial^b_\beta N^{(\beta)}_{ck})$

(g) $2C^{a\ b}_{\alpha k\beta} = -\delta_k g^{ba}_{\beta\alpha} + g^{b\ c}_{\beta(\alpha)}\partial^a_\alpha N^{(\alpha)}_{ck} + g^{a\ c}_{\alpha(\beta)}\partial^b_\beta N^{(\beta)}_{ck})$

(h) $2C^{abc}_{\alpha\beta\gamma} = \partial^c_\gamma g^{ab}_{\alpha\beta} + \partial^a_\alpha g^{bc}_{\beta\gamma} - \partial^b_\beta g^{ca}_{\gamma\alpha}.$

Proof. The proof follows from (7.10) and the condition $[g_{k\alpha}^{\ a}] = 0$ for $\forall\alpha = \overline{1,K}$ □.

It is obvious that in (7.11)

$$F^a_{\alpha ih} = -F_{i\alpha h}^{\ a} \quad C_{k\alpha\beta}^{\ ab} = -C^{a\ b}_{\alpha k\beta}.$$

In the recurrent K-Hamilton space in which $T_H(E^*)$ is orthogonal to $T_V(E^*)$ i.e. where the metric tensor has the property $[g_{i\alpha}^{\ a}] = 0$ for $\forall\alpha\overline{1,K}$, from (6.11) we get

(a) $F_{i\ k}^{\ h} = g^{hj} F_{ijh}$

(b) $0 = g^{\alpha\delta}_{ad} F_{i\alpha k}^{\ a}$

(c) $0 = g^{hi} F^a_{\alpha ik}$

(d) $\overset{(\alpha)}{F}{}^{a}_{\ dk} = g^{(\alpha)\beta}_{d\ b} F^{ab}_{\alpha\beta k}$

(7.12)

(e) $C_{i\ \gamma}^{\ hc} = g^{hj} C_{ij\gamma}^{\ \ c}$

(f) $0 = g_{da}^{\delta\alpha} C_{i\alpha\gamma}^{\ ac}$

(g) $0 = g^{hi} C_{\alpha i\gamma}^{a\ c}$

(h) $\overset{(\alpha)}{C}{}^{a\ c}_{\ d\gamma} = g^{(\alpha)\beta}_{\ d\ \ b} C^{abc}_{\alpha\beta\gamma}$.

The strongly distinguished connection coefficients which appear on the right-hand side of (7.12) for the torsion free, metric K-Hamilton space are determined by (7.11).

For the torsion free strongly distinguished connection for which ($\partial_\alpha^a N_{bk}^\beta = 0\ \forall\alpha, \beta = \overline{1, K}$ the relation (5.14) holds. These relations may be obtained for the above case also from (7.11) and (7.12).

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