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THE CHARACTER OF THE SOLUTION OF FIRST ORDER DIFFERENCE EQUATION IN THE FIELD OF MIKUSIŃSKI OPERATORS

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Abstract

In this paper a class of differential equations of the first order in the field of Mikusiński operators, $\mathcal{F}$, is considered. This class may correspond to a class of partial differential equations with constant coefficients.

We construct a discrete analogue for these differential equations in $\mathcal{F}$, and obtain difference equations of the first order. We construct the approximate solutions of such difference equations and treat them as the approximate solution of the corresponding partial differential equation. Also we estimate the error of approximation.

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1. Notations and notions

The set of continuous functions on $[0, \infty)$, denoted by $\mathcal{C}_+$, with the usual addition and the multiplication given by the convolution

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \quad t \geq 0,$$

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is a ring. By the Tichmarsh theorem, $\mathcal{C}_+$ has no divisors of zero, hence its quotient field can be defined (see [2]). The elements of this field, the Mikusiński operator field, $\mathcal{F}$, are called operators. They are quotients of the form

$$\frac{f}{g}, \quad f \in \mathcal{C}_+, \ 0 \not\equiv g \in \mathcal{C}_+,$$

where the last division is observed in the sense of convolution.

The most important operators are the integral operator l and its inverse operator, the differential operator, s. It holds

$$ls = I, \quad l^\alpha = \{\frac{t^{\alpha-1}}{\Gamma(\alpha)}\}, \quad \alpha > 0.$$

If the function $f \in \mathcal{C}_+$ has a continuous n-th derivative, then it holds

$$\{f^{(n)}(t)\} = s^n f - s^{n-1} f(0) - ... - f^{(n-1)}(0)I.$$

Let us denote by $\mathcal{F}_c$ the subset of $\mathcal{F}$ consisting of the operators representing continuous functions. If $a \in \mathcal{F}_c$ represents the continuous function $a(t), t \geq 0$, then we put $a = \{a(t)\}$. By $\mathcal{F}_I$ we denote the subset of $\mathcal{F}$ consisting of the elements αI, for some numerical constant α.

For two operators $a = \{a(t)\}$ and $b = \{b(t)\}$ from $\mathcal{F}_c$ the relation "$\leq$" is defined by (see [2])

$$a \leq b \quad \text{iff} \quad a(t) \leq b(t) \ \text{ for each } \ t \geq 0.$$

Analogously, we shall say for two operator functions $a(x)$ and $b(x)$ that

$$a(x) \leq_T b(x), \quad x \in [c, d],$$

if $a(x)$ and $b(x)$ are representing continuous real valued functions of two variables, $a(x) = \{a(x,t)\}$, $b(x) = \{b(x,t)\}$, and it holds

$$a(x,t) \leq b(x,t), \quad \text{for } \ t \in [0,T], \quad x \in [c,d].$$

2. Introduction

In this paper we consider and analyze the following problem in the field of Mikusiński operators

$$\sum_{m=0}^{p} k_m s^m u'(x) + \sum_{j=0}^{r} q_j s^j u(x) = f(x), \quad x \in \mathbf{R}, \tag{1}$$

with the initial condition

$$u(0) = R. \tag{2}$$

In relation (1) (and throughout the paper) we suppose that p and r are natural numbers, the coefficients k_m, $m = 0, 1, .., p$, and q_j, $j = 0, 1, ..., r$, are numerical constants, $k_p \neq 0, q_r \neq 0$, and $f(x)$ is a given, while $u(x)$ is the unknown operational function. In relation (2), R is a given operator. In this paper we shall mostly observe two cases: the first when R represents a continuous function (i.e. R is from $\mathcal{F}_c$), and the other when R is from $\mathcal{F}_I$. (In distribution theory, the last case would mean that R is a multiple of the Dirac delta measure.)

If $f(x)$ and $u(x)$ are operators representing continuous functions $f(x,t)$ and $u(x,t)$ respectively, then equation (1) corresponds to the partial differential equation

$$\sum_{m=0}^{p} k_m \frac{\partial^{m+1} u(x,t)}{\partial x \partial t^m} + \sum_{j=0}^{r} q_j \frac{\partial^j u(x,t)}{\partial t^j} = f_1(x,t), \tag{3}$$

with certain given conditions. In the last equation the right–hand side function $f_1(x,t)$ is expressed via the function $f(x,t)$ from (1) and the imposed conditions

$$\frac{\partial^j u(x,0)}{\partial t^j} = 0, \; j = 0, 1, .., r-1, \; x \in \mathbf{R},$$

(the Cauchy conditions), and

$$u(0,t) = R(t), \quad t \geq 0,$$

where $R = \{R(t)\}$.)

In the field of Mikusiński operators, the solution of the homogeneous equation for the differential equation (1) has the form $e^{x\omega}$, where ω is the solution of the characteristic equation of equation (1):

$$\sum_{m=0}^{p} k_m s^m \omega + \sum_{j=0}^{r} q_j s^j = 0.$$

It is well known that in the field of Mikusiński operators one can apply the algebraic operations (like, e.g., addition, multiplication, division, etc.) in the same way as when one deals with real numbers. The solution of the characteristic equation has the form

$$\omega = \frac{-\sum_{j=0}^{r} q_j s^j}{\sum_{m=0}^{p} k_m s^m} = \frac{Q}{k_p} \sum_{i=0}^{\infty} (-1)^{i+1} (\frac{P}{k_p})^i ..$$

where

(4)
$$P = \sum_{m=0}^{p-1} k_m l^{p-m} \quad \text{and} \quad Q = \sum_{j=0}^{r} q_j l^{p-j}.$$

The operator P is from $\mathcal{F}_c$. If $p \geq r$ the solution ω of the characteristic equation is logarithmic, i.e. the exponential function $e^{x\omega}$ exists (see [3], p.18). Then the solution of the homogeneous equation corresponding to the problem (1), (2), $R \cdot e^{x\omega}$, is from $\mathcal{F}_c$, provided that $R \in \mathcal{F}_c$ and $x \neq 0$.

In the case $p < r$, the solution of the homogeneous equation does not exist as an operator from the Mikusiński operator field.

The approximate solution of the characteristic equation can be taken in the form as in the paper [4]

$$\omega_N = \frac{Q}{k_p} \sum_{i=0}^{N} (-1)^{i+1} (\frac{P}{k_p})^i.$$

As is usual in numerical analysis, we replace the derivative $u'(x)$ with the quotient

$$\frac{u(x+h) - u(x)}{h}, \quad h > 0.$$

Denoting by

$$A = \sum_{m=0}^{p} k_m s^m \quad \text{and} \quad B = \sum_{j=0}^{r} q_j s^j,$$

we obtain the following difference equation in the field $\mathcal{F}$ (see [1], p.14):

(5)
$$A \frac{u(x+h)}{h} + (B - \frac{A}{h}) u(x) = f(x).$$

Put $x_0 = 0$, denote $x_n = x_{n-1} + h$, $n \in \mathbf{Z}$, and define the operator f_n by

$$f_n = f(x_n), \qquad n \in \mathbf{Z}.$$

Then instead of equation (5), we shall observe the following difference equation in $\mathcal{F}$:

(6) $$au_n + bu_{n+1} = f_n, \quad n \in \mathbf{Z},$$

where

(7) $$a = \sum_{j=0}^{r} q_j s^j - \frac{I}{h} \sum_{m=0}^{p} k_m s^m,$$

and

(8) $$b = \frac{I}{h} \sum_{m=0}^{p} k_m s^m.$$

We shall impose the condition $u_0 = R$, where the operator R is the one given in (2).

The formal solution of equation (6) has the form

(9) $$u_n = \sum_{k=-\infty}^{\infty} G_{n-k} f_k,$$

where

(10) $$G_{n-k} = \begin{cases} R(-\frac{a}{b})^{n-k}, & n-k \leq 0, \\ (R - \frac{I}{a})(-\frac{a}{b})^{n-k}, & n-k \geq 1, \end{cases}$$

and R is the operator given by the initial condition (2). In the paper [6], the exact solution of equation (6), u_n, $n \in \mathbf{Z}$, was treated as the approximate solution of the problem (1), (2).

It turns out that the obtained approximate solution, given as an infinite series, is inconvenient for computer calculating. For that reason, in this paper we shall construct and analyze the approximate solution $\tilde{u}_n$ of equation (6), taking finite sums instead of infinite sums.

The error of approximation will show that the solution $\tilde{u}_n$ can be treated as the approximate solution of the problem (1), (2). As a matter of fact, this approximate solution produces somewhat "greater" error, however the advantage of this method is that the the solution can be handled more easily.

3. The approximate solution

In the paper [6] it was proved

Lemma 1. *If in equation (1) it holds that $p < r$, i.e. $r = p + \nu$, $\nu \in \mathbf{N}$, and a and b are given by relations (7) and (8), then the operator $\frac{b}{a} \in \mathcal{F}_c$. However, operator $\frac{a}{b} \in \mathcal{F}$ belongs neither to $\mathcal{F}_c$ nor to $\mathcal{F}_I$, and it can be written as*

$$\begin{aligned} \delta : &= \frac{a}{b} \\ (11) \quad &= s^{r-p}\left(\frac{hq_r}{k_p}I + \frac{hQ_1 - P_1}{k_p} + \frac{hq_r + hQ_1 - P_1}{k_p}\sum_{i=1}^{\infty}(-1)^i(\frac{P}{k_p})^i\right). \\ &= s^{\nu}(\alpha_2 I + \beta_{c,2}), \end{aligned}$$

where α_2 is a nonzero numerical constant, while $\beta_{c,2}$ is an operator from $\mathcal{F}_c$.

Let us introduce the operator $\tilde{\delta}_N$

$$\begin{aligned} \tilde{\delta}_N : &= s^{r-p}\left(\frac{hq_r}{k_p}I + \frac{hQ_1 - P_1}{k_p} + \frac{hq_r + hQ_1 - P_1}{k_p}\sum_{i=1}^{N}(-1)^i(\frac{P}{k_p})^i\right). \quad (12) \\ &= s^{\nu}(\alpha_2 I + \tilde{\beta}_{c,2}^N), \end{aligned}$$

where $N \in \mathbf{N}$, and

$$(13) \qquad \tilde{\beta}_{c,2}^N = \frac{hQ_1 - P_1}{k_p} + \frac{hq_r + hQ_1 - P_1}{k_p}\sum_{i=1}^{N}(-1)^i(\frac{P}{k_p})^i.$$

Also, in paper [6] we proved

Theorem 1. *Assume that $r = p+\nu$ and that the operators a and b are given by (7) and (8), and, additionally, assume that the right-hand side operators f_n, $n \in \mathbf{Z}$, from relation (6) either*

I) satisfy the following equalities and estimates:

$$f_n = F_n I, \quad |F_n| < F, \quad n \in Z$$

II) are from $\mathcal{F}_c$ and satisfy the estimates

$$|f_n| \leq_T F_T l,$$

for some numerical constants $F_n, n \in \mathbf{Z}$ *and some constant* F, *resp.* F_T, *independent from* n.

Then there exists a solution of equation (6) in the field of Mikusiński operators and it represents a continuous function and has the form

$$
(14)\qquad
\begin{aligned}
u_n &= \frac{I}{a}\sum_{k=n}^{\infty}(-1)^{n-k}(\delta)^{k-n} f_k \\
&= \frac{I}{a}\sum_{k=n}^{\infty}(-1)^{n-k}(s^{\nu}(\alpha_2 I+\beta_{c,2}))^{k-n} f_k,
\end{aligned}
$$

where $\alpha_{@}$ *and* $\beta_{2,c}$ *are given in (11).*

Using conditions either I or II, we easily get that the series (14) converges in the field of Mikusiński operators and represents a continuous function. Since the operator $\frac{I}{a}$ is from $\mathcal{F}_c$, the solution given by relation (14) represents a continuous function.

Since $R = 0$ the operator $\frac{a}{b}$ is neither an operator from $\mathcal{F}_c$, nor from $\mathcal{F}_I$, then in the case the operator G_{n-k} will not represent a continuous function nor it will be from $\mathcal{F}_I$. In the field $\mathcal{F}$ we do not consider such series.

Now, we shall construct the approximate solution of difference equation (6) for the case when $r > p$:

$$
(15)\qquad
\begin{aligned}
(\tilde{u}_n)_{N,M} &= \frac{I}{a}\sum_{k=n}^{M}(-1)^{n-k}\left(\tilde{\delta}_N\right)^{k-n} f_k \\
&= \frac{I}{a}\sum_{k=n}^{M}(-1)^{n-k}\left(s^{\nu}(\alpha_2 I+\tilde{\beta}_{c,2}^{N})\right)^{k-n} f_k.
\end{aligned}
$$

Let us remark that the approximate solution given by relation (15) is of the same form as the exact solution. This means that it also represent continuous function.

4. The error of approximation

In view of Theorem 2, we shall give the error of approximation for the solution of the problem (1), (2) for $r > p$. This solution is approximated by the approximate solution $\tilde{u}_n$ of the difference equation (6) which exact solution is u_n.

Theorem 2. *Let us suppose that the first and the second derivative of the function $u(x)$ (which is the solution of equation (1)) are continuous operational functions.*

If we denote by $u(x_n)$ the value of the exact solution of equation (1) at the point $x = x_n$, then the error of approximation for its approximate solution obtained as the approximate solution of equation (6) $\tilde{u}_n$, can be estimated by

$$|u(x_n) - \tilde{u}_n| \leq_T h\mathcal{R}_\infty(T)M_2(X,T)l + R_2(T)l \tag{16}$$

where $\mathcal{R}_\infty(T)$ and

$$M_2(X,T) = \max_{(0\leq t\leq T)\times(-X\leq x\leq X)} |\frac{\partial^2 u(x,t)}{\partial x^2}|.$$

are positive numerical constants.

Proof. From the difference between the following equations

$$\sum_{m=0}^{p} k_m s^m u'(x_n) + \sum_{n=0}^{r} q_j s^j u(x_n) = f(x_n),$$

$$\sum_{m=0}^{p} k_m s^m \frac{u_{n+1} - u_n}{h} + \sum_{j=0}^{r} q_j s^j u_n = f(x_n),$$

we obtain

$$\sum_{j=0}^{r} q_j s^j (u(x_n) - u_n) = -\sum_{m=0}^{p} k_m s^m (u'(x_n) - \frac{u_{n+1} - u_n}{h}),$$

wherefrom we have

$$|u(x_n) - u_n| \leq | \frac{-\sum_{m=0}^{p} k_m l^{r-m}}{\sum_{j=0}^{r} q_j l^{r-j}} | \cdot | u'(x_n) - \frac{u_{n+1} - u_n}{h} | . \tag{17}$$

In [6] we obtain

$$|u(x_n) - u_n| \leq_T h\mathcal{P}_1(1 + \frac{T}{2}\mathcal{Q}_1(T)exp(\mathcal{Q}_1(T), T))M_2(X,T)Tl,$$

where the numerical constants $\mathcal{P}_1(T)$ and $\mathcal{Q}_1(T)$ are obtained from

$$\left\{ \frac{|\sum_{m=0}^{p} k_m \frac{t^{r-m-1}}{(r-m-1)!}|}{|q_r|} \right\} \leq \mathcal{P}_1(T)l,$$

and

$$\left\{ \frac{|\sum_{j=0}^{r-1} q_j \frac{t^{r-j-1}}{(r-j-1)!}|}{|q_r|} \right\} \leq \mathcal{Q}_1(T)l.$$

Hence

$$(18) \quad |u'(x_n) - \frac{u_{n+1} - u_n}{h}| \leq_T M_2(X,T)lh, \qquad x_n \in [-X, X] \quad X > 0.$$

The error of approximation is

$$|u(x_n) - \tilde{u}_n| \ = \ |u(x_n) - u_n + u_n - \tilde{u}_n| \leq |u(x_n) - u_n| + |u_n - \tilde{u}_n| \ .$$

Using the estimation

$$|u_n - \tilde{u}_n| \leq_T R_2(T)l,$$

we obtain the relation (16). The last estimation could be done because in [6] it was proved that the operator u_n represented a continuous function for each n. Therefore the operator $\tilde{u}_n$ is from $\mathcal{F}_c$, also.

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