The work of professor Djuro Kurepa in the set theory and the number theory *

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1. Introductory word

On occassions like this it is almost impossible to realize in detail the grand, exceptionally full, diverse and rich creative work of academian Djuro Kurepa, even in some of its segments. Djuro Kurepa, a man of wide general culture, one of those rare mathematical zealous workers popularly called "encyclopedists", left a deep trail in almost any branch of mathematics. He impressed by his deep understanding of problems, finding out the "proper" ones, and by his great intuition. He knew how to guess and set hypothesis on which the pleiads of mathematicians worked and have been working, and whose solving often demanded quite new methods. Because of all that, these few observations of mine should be understood as a wish to pay tribute to professor Kurepa rather than to provide a bit more complete review of his creativity in the set and number theories. Really, for such a thing another "encyclopedist" would be needed.

2. Suslin's and Cantor's problem of the continuum

In his doctoral dissertation *Ensembles ordonnés at ramifiés*, defended at the Sorbonne in the year 1935, professor Kurepa introduced for the first time the notion of *partially well-ordered sets* (in French *tableaux ramifiés*). In order the "story" could be followed more easily, we are giving here only some necessary definitions from his book ([12]), though sometimes with a little bit obsolete lexicon and notations.

Definition 2.1. *Partially ordered set is a triple* $(S, \approx, \leq)$*, where* S *is a set,* $\approx$ *an equivalence relation and* $\leq$ *a binary relation on* S*, such that the following conditions are fulfilled:*

(reflexivity): $x \approx x$, $x \leq x$;
(transitivity): if $x \approx y \approx z$ *then*2±1$\approx z$, *if* $x \leq y \leq z$ *then* $x \leq z$;
(partial symmetry): if $x \approx y$ *then* $y \approx x$ *and* $x \leq y$ *and* $y \leq x$.

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$x = y$ is an abbrevation for $x \leq y \leq x$.

$x < y$ is to replace: $x \leq y$ and $x \neq y$.

If $\approx$ is just the identity relation $\equiv$, it is written only $(S, \leq)$ instead of $(S, \equiv, \leq)$, and in the further text such partially ordered sets will be mainly considered. The intervals for the given partial ordering are defined in a standard way. So, for instance, for $x \in S$, it is $(-\infty, x)_S = \{y \in S \mid y < x\}$, $[x, \infty)_S = \{y \in S \mid x \leq y\}$.

Definition 2.2. $[x]_S = (-\infty, x, \infty)_S = (-\infty, x]_S \cup [x, \infty)_S$ *is called the origin (descent) of the element* x *with respect to the partial ordering* S.

The left knot of the element $x \in S$, *in notation* $(-\infty, x|_S$, *is the set* $\{y \in S \mid (-\infty, x)_S = (-\infty, y)_S$. *Analogously is defined the right knot of* x.

The initial (final) section of the set S *is a subset of* S, X, *for which it holds: if* $x \in X$ *then* $(-\infty, x] \subseteq X$ $([x, \infty) \subseteq X)$.

A filter of the set X *is a proper initial section of the (partially ordered) set* $(P(X), \supseteq)$.

A partially ordered set $(S, \approx, \leq)$ *is ordered from below or from the left (from above or from the right) iff it satisfies the condition of ramification:*

$\forall x \in S$ $(-\infty, x]$ $([x, \infty))$ *is totally ordered (any two elements are comparable).*

A mutually ordered set is a set ordered both from the left and from the right.

A half-ordered set is a set which is ordered either from the left or from the right, or both from the left and the right.

A partially ordered set $(S, \leq)$ *is partially well-ordered iff it satisfies the condition:*

(D) any totally ordered part of S *is well-ordered.*

For the sake of convenience, any partially well-ordered set which is also ordered from the left will be also called partially well-ordered.

A significant part of the results of professor Kurepa is related just to establishing a connection between the properties of partially well-ordered sets and Cantor's problem, that is between the properties of partially well-ordered sets and Suslin's problem of the continuum. More concretely about that after introducing some new definitions.

Definition 2.3. *For a partially ordered set* $(S, \leq)$ *it is:*

$k_c S \overset{\text{def}}{=} sup\{|x| \mid x \text{ is a well-ordered subset of the set } S\}$

the index c is coming from the Latin verb cresco (-ēvi, ētum) (= increase); let us say at this point that professor Kurepa denoted the cardinality of a set A by kA;

$k_d S \overset{\text{def}}{=} sup\{|x| \mid x$ *is well-ordered with respect to the relation* $\geq$ *subset of* $S\}$

we have: $k_d(S, \leq) = k_c(S, \geq)$; *d is coming from decresco (= decrease);*

$k_s S \overset{\text{def}}{=} sup\{|x| \mid x$ *is a totally unordered subset (no two of its different elements are comparable) of* $S\}$

s is from Suslin;

$bS \overset{\text{def}}{=} sup\{|x| \mid x$ *is a mutually ordered subset of* $S\}$.

It holds that the cardinal bS is equal to the cardinal: $b_1 S \overset{\text{def}}{=} sup\{|x| \mid x$ is either a totally ordered or totally unordered subset of $S\}$.

Definition 2.4. *If it is, for* $x \in S$, $(-\infty, x) = \emptyset\ (= \nu)$, *then x is the initial element of the set S;*

$RS = R_0 S \overset{\text{def}}{=} \{x \mid x$ *is the initial element of* $S\}$ *is the initial row (sort, group, category) of the set S;*

In general:

$$R_\alpha S \overset{\text{def}}{=} R_0(S \setminus \bigcup_{\xi<\alpha} R_\xi)$$

and

$\gamma S = \gamma(S) \overset{\text{def}}{=}$ *the least ordinal* α *for which it is* $R_\alpha S = \emptyset$ *is the so-called rank of the ordered set S.*

$\gamma(x) = \gamma(x; S) \overset{\text{def}}{=}$ *the least ordinal* α *for which it is* $x \in R_\alpha S$ *is the rank of the element x.*

The following problem, inspired maybe by one not quite reality-based problem from biology (considering the descendants of microbes which self-propagate), initiated a lot of others, even today very actual problems in set theory:

if T is partially well-ordered set (which here means that for all $a \in T$ *the interval* $(-\infty, a]$ *is well-ordered) for which it holds:*

(1) *for all* $a \in T$ *it is* $\gamma(-\infty, a, \infty) = \gamma[a] = \omega_1$;

and

(2) *for all* $a \in T$ *it is* $|R_0(a, \infty)| \geq 2$ *(*$R_0(a, \infty)$ *is the set of all immediate successors (descedants) of the element a),*

whether it is possible to choose from any row of the second sort $R_{\omega\alpha}T$, $\alpha < \omega_1$) *one element* $a_{\omega\alpha}$ ($\in R_{\omega\alpha}T$), *so that the set of the chosen elements is a totally unordered subset of* T, *i.e. that for* $\alpha < \beta < \omega_1$ *it is neither* $a_{\omega\alpha} \leq a_{\omega\beta}$ *nor* $a_{\omega\alpha} > a_{\omega\beta}$?

Of course, the difficult case is the one when each row and each ordered part of the set T is at most countable. N. Aronszajn was actually the first to prove that such case can appear – about it a bit later. Of the special interest is the fact that the question of the existence of the function $a_{\omega\alpha} \in R_{\omega\alpha}$ is equivalent to Suslin's problem:

does a dense totally ordered set, in which there are no uncountable many disjoint intervals, necessarily have a realization inside the linear continuum?

Namely, we have

Teorema 2.5. *Dj. Kurepa (Basic Theorem). The answer to Suslin's problem is positive iff, for any partially well-ordered set* T, *from* $bT \leq \aleph_0$ *it follows* $|T| \leq \aleph_0$.

Whence

Teorema 2.6. *The answer to Suslin's problem is positive iff each partially well-ordered uncountable infinite set contains an uncountable part which is either totally ordered or totally unordered.*

N. Aronszajn, who collaborated with professor Kurepa, came in the year 1934 to the example:

there exists a half-ordered infinite set A with the properties:

$\gamma A = \omega_1$;

$|R_\alpha A| \leq \aleph_0$ *za svako* $\alpha < \omega_1$)

and

$k_c A \leq \aleph_0$,

which was firstly published in Kurepa's thesis. But, let us have a look at the basic theorem. It sais in fact, in the contemporary language of set theory:

There exists a Suslin line iff there exists a Suslin tree,

that is, let us be more precise:

There exists no a Suslin line iff there exists no a Suslin tree.

Let us recall:

A *Suslin line* is a dense linearly ordered set which satisfies *countable chain condition* or, shortly, *c.c.c*, and which is not *separable*, i.e. it does not have dense countable subset. *Suslin's hypothesis* (SH) is the statement:

there exists no Suslin line.

A tree is just a partially well-ordered set, ordered from the left, and this term has completely removed Kurepa's expression. Partially because of it, I assume, the new generations of mathematicians are not always quite aware of the great significance of the pioneering work of professor Kurepa in this field. I also allow the possibility that this change of the terms was caused by the predominance of the new "graphic representation". Namely, professor Kurepa "laid down" the partially well-ordered sets horizontally – see, for instance, Figure 17.2.1 in [12], while, let us say, G. Birkoff put the initial level horizontally, and the rest of the set beneath it. If one rotates his figure for ninety degrees (in the positive direction – otherwise one would obtain Birkoff's figure), it "appears a tree" or, in general case ($|R_0S| > 1$), "many trees" (the condition that a tree has only one root is a part of the definition of, for example, *normal trees*).

Of course, since it is a word about trees, the other terms are "adapted" as well. So, for instance, if T is a tree and $x \in T$, then we are talking about the *height* of the element x (in notation usually $h(x,T)$) rather than about its rank; the *height of a tree* is again the rank; a branch of a tree is a maximal linearly ordered subset of the tree; an *antichain* is is totally unordered subset of the tree.

The Suslin tree is a tree of the height ω_1, whose each branch is at most countable and whose each antichain is at most countable.

The Aronszajn tree is exactly the above given example.

Let us still note that the Suslin problem in the original form (Fundamenta Mathematica,1/1920) states:

is every dense, unbounded and complete (any subset bounded from above has the supremum) linearly ordering isomorphic to the real line?

In the book of W. Sierpiński, "Nombres transfinis" (Paris, 1926) this problem was characterized as a very difficult one.

Today it is known that this problem is independent of ZFC; in other words, both $ZFC + SH$ and $ZFC + \neg SH$ are consistent theories.

For a better understanding of the Suslin and related problems it is of use the following Kurepa's result ([11])

Teorema 2.7. *Each uncountable partially well-ordered set for which there exists a strongly increasing (decreasing) real function contains an uncountable totally unordered subset.*

In his papers professor Kurepa has considered one more general Suslin's problem. The word is about the following.

For a totally ordered set S let it be:

$k_1 S = k_1(S) \stackrel{\text{def}}{=} inf\{|E| \mid E$ is (allwhere) dense subset of the set $S\}$;

$k_2 S = k_2(S) \stackrel{\text{def}}{=} sup\{|F| \mid F$ is nonempty family of disjoint intervals of the set $S\}$.

The function $k_2 S$ and its "more general version" ($- k_s S$), today known as *cellularity* (of partially ordered sets, Boolean algebras, topological spaces), were defined for the first time in Kurepa's thesis. I presume that he was the "godfather" at least to some of the mentioned ones and certainly to some about which there will be no word this time.

Generally, for infinite totally ordered sets it holds:

$$k_1 S = k_2 S \quad \text{or} \quad k_1 S = (k_2 S)^+.$$

The general Suslin problem is searching for the answer to the question:

does for any infinite totally ordered set S it hold $k_1 S = k_2 S$?

On the other hand, this is equivalent, in the sense that the answer to the both questions is the same, being either affirmative or negative, to:

does for any half-ordered infinite set T it hold: $bT = |T|$?

Again, it is known as well that there are no other cardinals between bT and $|T|$. In his thesis professor Kurepa gave twelve equivalent statements from which it follows the positive answer to the Suslin problem, but there are no necessarily the implications in the opposite direction. We adduce now only one of them – the *ramification hypothesis* – *RH*, with the note that *degenerated* means *mutually ordered*:

for every tree $(T, \leq)$ there is a degenerated subtree of the cardinality $b(T, \leq)$, i.e. the number $b(T, \leq)$ is reached in $(T, \leq)$.

or, in the original version:

if T is a partially (half-) well-ordered set, then the supremum bT is reached in T, that is there exists mutually ordered part of T whose cardinal number is exactly bT.

As for cellularity, the following Kurepa's results are of the special interest.

Teorema 2.8. *If L is a Suslin line then $k_2(L \times L) \geq \aleph_1$.*

Teorema 2.9. *If X_i, $i \in I$, are topological spaces of the cellularity $\leq \kappa$ then the cellularity of $\prod_{i \in I} X_i$ is $\leq 2^\kappa$.*

The first of these theorems together with the assertion

if $MA(\omega_1)$ holds then the product (of arbitrary cardinality) of the spaces satisfying c.c. condition also satisfies this condition

implies:

Teorema 2.10. $ZFC + MA(\omega_1) \vdash SH$,

where, in general, for a given cardinal κ, $MA(\kappa)$ (the so-called Martin's axiom – for κ) is the condition:

if $(P, \leq)$ is a partially ordered set with the c.c. condition (i.e. of the cellularity ω) and $\mathcal{D}$ a family of $\leq \kappa$ dense subsets of P, then there exists a filter G in P which has a nonempty intersection with any member of the family $\mathcal{D}$;

$D \subset P$ is dense iff $\forall p \in P \, \exists d \in D \;\; d \leq p$; $G \subset P$ is a filter iff it holds: $\forall p, q \in G \, \exists r \in G \;\; r \leq p \wedge r \leq q$ and (2) $\forall p \in G \, \forall q \in P \;\; p \leq q \implies q \in G$. Of course, if it holds $MA(\omega_1)$, then $2^\omega > \omega_1$ (for it does not hold $MA(2^\omega)$).

Besides, by defining the ω_1-Suslin tree, professor Kurepa "planted" his own tree, presented in any "set forest".

Definition 2.11. *A path of a κ-tree T (T is a tree of the height κ) is a branch which interects each level $Lev_\alpha(T) = \{x \in T \mid h(x, T) = \alpha\}$, $\alpha < \kappa$.*

For any regular cardinal κ, κ-Kurepa tree is a κ-tree with κ^+ paths.

κ-KH is the assertion "there exists κ-Kurepa tree". Kurepa's Hypothesis, shortly KH, is ω_1-KH.

$\mathcal{F} \subseteq P(\kappa)$ is κ-Kurepa family iff it holds: $|\mathcal{F}| \geq \kappa^+$ and $\forall \alpha \leq \kappa \;\; |\{A \cap \alpha \mid A \in \mathcal{F}\}| < \kappa$.

Let us note that in his book, [1], F. Drake sais that it seems that Kurepa himself took in fact for the hypothesis $\neg KH$.

Professor Kurepa proved concretly in [9], again said in the words of the new terminology:

Teorema 2.12. *KH if and only if there exists ω_1-Kurepa family.*

This proposition holds generally for any regular cardinal. Today, it is known, let us mention only:

$V = L[X] \vdash KH$ $(\neg SH)$, where $X \subseteq \omega_1$, in particular, $V = L \vdash KH$, consequently, $ZFC + KH$ is consistent (on condition that ZFC is consistent;

$\diamondsuit^+ \vdash KH$, where $\diamondsuit^+$ is the statement: there exists a family $\mathcal{F} \subseteq P(\omega_1)$, such that it holds: $\forall \alpha < \omega_1 \ |\{X \cap \alpha \mid X \in \mathcal{F}\}| \leq \omega$ and $\forall A \subseteq \omega_1 \ |A| = \omega_1 \Longrightarrow \exists X \in \mathcal{F}\,(|X| = \omega_1 \wedge X \subseteq A)$.

$ZFC + \neg KH$ is consistent iff $ZFC+$ *there exists strongly inaccessible cardinal* is consistent (κ is strongly inaccessible iff it is a regular cardinal greater than $\aleph_0$ and for, each $\lambda < \kappa$, $2^\lambda < \kappa$).

To a great extent professor Kurepa devoted his research work to Cantor's Hypothesis, better known as *Continuum Hypothesis* (CH): $2^{\aleph_0} = \aleph_1$, connected it with partially well-ordered sets. From the following two lemmas:

Lemma 2.13. *For any partially well-ordered set W it holds:*

$$|W| \leq (2k_sW)^{k_cW},$$

Lemma 2.14. *For any infinite partially well-oredred set W it holds: $|W| \leq 2^{bW}$,*

it follows the so-called *Basic Double Theorem*

Teorema 2.15. *If a partially well-ordered set W satisfies the conditions: $bW \leq \aleph_0$ and the cardinal number of each of its row (level) is less than $\aleph_0$ (less than or equal to $\aleph_0$), then $\gamma W < \omega_1$ $(\gamma W < \omega_{\nu(0)+1})$, where $2^{\aleph_0} = \aleph_{\nu(0)})$. Besides, ω_1 $(\omega_{\nu(0)+1})$ is the least ordinal which always satisfies the last relation;*

and its corollary

Corollary 2.16. *Cantor's Hypothesis holds if and only if from $bW \leq \aleph_0$ follows:*

$$|sup\{\gamma W \mid |R_\alpha W| < \aleph_0,\, \alpha < \gamma W\}| = sup\{|\gamma W| \mid |R_\alpha W| \leq \aleph_0,\, \alpha < \gamma W\}.$$

It is also of interest the following result which includes "in the play" the partially ordered set (C), whose elements are the sequences of natural numbers, finite or countable infinite, ordered by the "initial coincidence".

Teorema 2.17. *Cantor's Hypothesis is equivalent to: the set* $((C), \subseteq)$ *contains a part* M *such that it is satisfied:*
$|M \cap R_\alpha(C)| \leq 1, \alpha < \gamma(C)$ *and* $\bigcup_{a\in M}[a]_{(C)} = (C)$.

By the way, let us notice that without Cantor's Hypothesis we would not be able to prove: $\aleph_1^{\aleph_0} = \aleph_1$.

In the year 1953 professor Kurepa announced in the French Academy at that time a revolutionary hypothesis – freely speaking, $2^{\aleph_0}$ can "jump over" any cardinal. (On the basis of a conversation with professor Kurepa, I concluded that the hypothesis itself was much older, which the whole thing makes even "more spectacular".) Almost twenty years later W. B. Easton confirmed this hypothesis ([2]); only the most basic conditions set by the very definition of cardinal exponentiation and König's lemma must be fulfilled in choosing an ordinal α for which it will hold in a suitable model $2^{\aleph_0} = \aleph_\alpha$ (more precisely, this assertion holds for any power 2^λ, λ – regular cardinal).

Speaking of the cardinal exponentiation, let us mention one interesting generalization of the relation $2^{\aleph_\alpha} = |\nu|^{\aleph_\alpha}$ for all ν, $1 < \nu \leq \aleph_\alpha$.

For an arbitrary set C and an ordinal α, let $C(\alpha)$ be the lexicographic linear ordering whose elements are the α-sequences with the members from C. Then it holds ([17])

Teorema 2.18. *For any* ν, $1 < \nu < \omega_\alpha$, *the chains* $2(\omega_\alpha)$ *and* $\nu(\omega_\alpha)$ *are of the same order type (that is they are isomorphic)*

From the same article we quote the following often applicable result.

Teorema 2.19. *For a given set* M *and a chain* C, *the following conditions are equivalent:*

(1) *the chain* C *is isomorphic to a subset of the power set* $P(M)$ *ordered by inclusion;*

(2) $k_1 C \leq |M|$ *i* $|\{x \in C \mid x$ *is unilateral limit point of the chain* $C\}| \leq |M|$.

3. The Axiom of Choice

The Axiom of Choice, in its most standard version:

if $\mathcal{F} = \{X_i \mid i \in I\}$ is an infinite family of nonempty mutually disjoint sets, then there exists a set, X, such that $X \subseteq \bigcup_{i \in I} X_i$ and, for each $i \in I$, $|X \cap X_i| = 1$,

is un-natural in comparison with the axioms of ZF-theory in the sense that the set (X), whose existence it "guarantees", remains "unknown" (undetermined). On the other hand, it looks so "natural" that it was difficult to notice it. It seems that G. Peano was the first who "stumbled upon" this axiom in a proof of an existential theorem of the ordinary differential equations. In the year 1904 E. Zermelo proved that the above statement is equivalent to the assertion that any set can be well-ordered – in fact, judging by the title of his article "Beweis, das jede Menge wohlgeordnet werden kann" (Math. Ann. 59, 514-516), at that time he himself did not have doubts about it. The behavior of the people fits into the "story" – once a man hears about it, he sees it everywhere. Today hundreds of either its equivalences or stronger or weaker versions (which is very often the question of the set or class theory in which one works) are known. Professor Kurepa significantly contributed to this subject, partly by solving some problems, partly by raising some "unpleasant" questions. *Inter alia*, he formulated the *Antichain Principle* (equivalent to the Axiom of Choice in ZF-theory):

(A) *Each partially ordered set has a maximal antichain, that is a maximal subset of mutually in-comparable elements.*

One of his important results is the following

Teorema 3.1. *The following propositions are equivalent (in ZF^0):*

(1) *The Axiom of Choice;*

(2) *Each set has a $\subseteq$-maximal subset (maximal with respect to the inclusion) with the property that the intersection of any two of its elements is nonempty;*

(3) *Each set has a $\subseteq$-maximal subset with the property that for any two its elements x, y it holds: $x \not\subseteq y \wedge y \not\subseteq x \wedge x \cap y \neq \emptyset$;*

(4) *Each set has a $\subseteq$-maximal subset with the property that for any two its elements x, y it holds: $x \subseteq y \vee y \subseteq x \vee x \cap y = \emptyset$.*

An analogous assertion holds also in the Neumann-Bernays-Gödel system without the Axiom of Regularity, shortly NBG^0 (which is the conservative extension of the system $ZF^0 = ZF-$ *the Axiom of Regularity*).

As far as this theorem is concerned, it should be mentioned that professor Kurepa was first to define the above properties, as well as many others, about which there will be here no word, but which are included in a lot of interesting results dealing with the Axiom of Choice.

Let us give also this theorem of professor Kurepa.

Teorema 3.2. *(ZF) The Axiom of Choice is equivalent to the statement: any set can be linearly ordered and any set has a $\subseteq$-maximal subset with the property that for any two of its different elements x, y it holds: $x \not\subseteq y \wedge y \not\subseteq x$.*

4. Kurepa's hypothesis in the number theory

As for the number theory, we shall focus our attention only on two Kurepa's hypothesis, with his own remark that "the science about spaces, that is topology is not the real (proper) place for the numbers" – [12] (but this is already a subject for another talk).

In the paper [20] Kurepa defined for the positive natural numbers *the left factorial*: $!n = \sum_{i=0}^{n-1} i!$ and asked the question: does it hold, for every natural $n \geq 2$:

$$(!n, n!) = 2?$$

His personal hypothesis was that the answer is positive. We shall denote it here by $KH1$. In the same paper professor Kurepa (among other things) proved

Teorema 4.1. $KH1$ *iff* $\forall n > 2 \quad !n \not\equiv 0 \ (mod\ n)$ *iff* $\forall p \in P \ (p > 2 \implies (!p, p) = 1)$ *(where P is the set of primes).*

This Kurepa's hypothesis is the only problem from the number theory of the Yugoslav mathematicians included in R. Guy's book *Unsolved Problems in Number theory.* Professor Žarko Mijajlović proved by using computer that $KH1$ holds for all natural numbers $n < 311009$ (after all, solving of this problem occupies the attention of many mathematicians). Otherwise, in the year 1991, Kurepa informed professor Mijajlović that he solved the problem and even announced a paper about it for the "Publications de l'Institute Mathématique", but for what reason he had not done it, it will unfortunately remain unknown.

The other Kurepa's hypothesis (again from [20]) is worded like this:

($KH2$) *the relation $m^2 \mid\, !n$ does not hold for all natural numbers greater than* 1 *with the exception of the case* $2^2 \mid\, !3$.

Professor Kurepa proved this assertion for $2 \leq m \leq 8$. Meanwhile the new "strengthenings" have appeared (see [22]), but, as in the case of the first hypothesis, the final solution is still not coming in sight.

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