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ON THE PRODUCT AND THE CONVOLUTION OF ULTRADISTRIBUTIONS

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Abstract

We consider the product and the convolution of ultradistributions by introducing some conditions from the microlocal viewpoint which is useful for the analysis of partial differential equations.

1. Introduction

In this paper we will study the basic operations for ultradistributions from the similar microlocal viewpoint of the hyperfunction theory as in M.Sato et al. [6]. One of the advantages that we have is that we can use a partition of unity in case of non-quasi-analytic ultradistributions.

However we need some estimates to evaluate the growth order there. We will treat the product, the integration, and the convolution of ultrdistributions. These are old matters, but the author consider it significant to review the theory from the new viewpoint. He would like to express his gratitude to the University of Novi Sad.

2. Definitions

For the theory of ultradistributions we review some conditions according to H. Komatsu [3] and others. Note that these conditions were first studied by C. Roumieu [5].

Let M_p be a sequence of positive numbers satisfying the following conditions:

$(M.0)$ $M_0 = M_1 = 1$;

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}, p = 1, 2, \ldots,$;

$(M.2)$ $\dfrac{M_p}{M_q M_{p-q}} \leq AB^p, 0 \leq q \leq p$ for some $A > 0$ and $B > 0$:

$(M.3)'$ $\sum_{p=1}^{\infty} \dfrac{M_{p-1}}{M_p} < \infty.$

Definition 1. *A function $f(x)$ on an open set U in $\mathbf{R}^n$ is called an ultradifferentable function of class (M_p) (resp. $\{M_p\}$) if for any compact set K in U and for $h > 0$ there is $C > 0$ (resp. if for any compact set K in U there are $h > 0$ and $C > 0$) such that*

$$sup_{x \in K} |D^\alpha f(x)| \leq C h^{|\alpha|} M_{|\alpha|}$$

Here

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \; a = (\alpha_1, \ldots, \alpha_n).$$

We denote by $\mathcal{D}f^{(M_p)}(U)$ (resp. $\mathcal{D}f^{M_p)}(U)$) the space of all the ultradifferentiable functions of class (M_p) on U (resp. $(\{M_p\})$). From now on $*$ will stand for either (M_p) of $\{M_p\}$.

Definition 2. *We denote by $Df_c^*(U)$ the space of ultradifferentiable functions of class $*$ with compact support in U. Refer to H. Komatsu [3] and others for its topology. We define the space $\mathcal{D}b^*(U)$ of ultradistribution of class $*$ by the strong dual space of $\mathcal{D}f_c^*(U)$.*

Note that the presheaves

$$U \mapsto \mathcal{D}f^*(U), U \mapsto \mathcal{D}b^*(U)$$

form sheaves. These sheaves are known to be fine.

Definition 3. *We denote by $\mathcal{B}(U)$ the space of Sato's hyperfunctions on an open set Ω in $\mathbf{R}^n$. We do not give a formal definition by using relative cohomology groups.*

We know that

$$U \mapsto \mathcal{B}(U)$$

forms a sheaf, and that this sheaf is flabby. Refer to M. Sato et al. [6] for more details. There exist injections.

$$\mathcal{D}f^*(U) \hookrightarrow \mathcal{D}b^*(U) \hookrightarrow \mathcal{B}(U).$$

We know that a hyperfunction f on U can be written as a formal sum of boundary values of holomorphic function $E_j(z)$ defined on infinitesimal wedges $U + i\Gamma_j 0$:

$$f(x) = \sum_{j=1}^{N} F_j(x + i\Gamma_j 0).$$

For an open set U in $\mathbf{R}^n$, we denote by $S^*U(\cong U \times S^{n-1})$ the cosphere boundle of U. We also denote by $T^*U(\cong U \times \mathbf{R}^n)$ the cotangent budle of U.

Let V be the complexification of U. Then $S^*_U V$ (resp. $T^*_U V$) can be identified with $\sqrt{-1}S^*U$ (resp. $\sqrt{-1}T^*U$).

Definition 4. *Let $f(x) \in \mathcal{B}(U)$. It is said to be micro-analytic at the point $(x, \sqrt{-1}, \xi\infty) \in \sqrt{-1}S^*U$ if, for a suitable above representation of $f(x)$ on a neighbourhood of $x, \Gamma_j \bigcap \{y \in \mathbf{R}^n; < \xi, y >> 0\} = \emptyset$ holds for all j.*

The set of all points at which f not micro - analytic is called the singular spectrum of f, and is denoted by $SS(f)$.

We remark that L. Hörmander [2] introduced the analytic wave front set WF_A which is proved to be identical with SS.

Definition 5. *Let $u(x) \in Db^*(U)$ with compact support. We introduce the wave front set $WF_A(u)$ in $\sqrt{-1}S^*U$. Let $\overset{\circ}{b} = (\overset{\circ}{x}, \sqrt{-1}\,\overset{\circ}{\xi}\,\infty) \in \sqrt{-1}S^*U$. Then $\overset{\circ}{b} \notin WF_*(u)$ if* *i) in case* $* = (M_p)$, *there exist a neighbourhood* V *of* $\overset{\circ}{x}$ *and a conic neighbourhood* Ξ *of* $\overset{\circ}{\xi}$ *such that for any* $\phi \in Df^*(U)$ *and any* $\epsilon > 0$ *there exists* $C_\epsilon > 0$ *satisfying*

$$|\widehat{\phi u}(\xi) \leq C_\epsilon \exp\{-M(\epsilon|\xi|)\}, \;\; \xi \in \Xi,$$

ii) in case $* = \{M_p\}$, *there exist a neighbourhood* U *of* $\overset{\circ}{x}$ *and a conic neighbourhood* Ξ *of* $\overset{\circ}{\xi}$ *such that for any* $\phi \in \mathcal{D}f^*(U)$ *there exist* $L > 0$ *and* $C > 0$ *satisfying*

$$|\widehat{\phi u}(\xi)| \leq C \exp\{-M(L|\xi|)\}, \xi \in \Xi.$$

Here we define the Fourier transform

$$\widehat{\phi u}(\xi) = \int_{\mathbf{R}^n} \phi(x) u(x) e^{\sqrt{-1}x\xi} dx,$$

and the associated function of M_p by

$$M(p) = \sup_p \log \frac{\rho^p M_0}{M_p}.$$

3. Theorems

First, we consider the product of ultradistributions.

Theorem 1. *Let* $u(x)$ *and* $v(x)$ *be ultradistributions of clase* $*$ *on an open* U *of* $\mathbf{R}^n$ *such that*

$$WF_*(u) \cap (WF_*(v))^\alpha = \emptyset.$$

Then the product

$$u(x)v(x) \in \mathcal{D}b^*(U)$$

exists with the property:

$$WF_*(uv) \subset \{(x, \sqrt{-1}(\theta\xi_1 + (1-\theta)\xi_2)\infty) | \\ (x, \sqrt{-1}\xi_1\infty) \in WF_*(u), (x, \sqrt{-1}\xi_2\infty) \in WF_*(v), \\ 0 \leq \theta \leq 1\}.$$

Here a: $\sqrt{-1}S^*(U) \to \sqrt{-1}S^*U$ *is the antipodal mapping (i.e.* $a(x, \sqrt{-1}\xi\infty) = (x, -\sqrt{-1}\xi\infty)$), *and* $(WF_*(v))^\alpha$ *denotes the image of* $WF_*(v)$ *by a.*

Proof. The theorem can be reduced to the case where u and v have small compact support in U. Moreover, we may assume from the beginning that there exist closed convex cones ξ_1 and Ξ_2 of $\mathbf{R}^n$ such that

$$WF_*(u_1) \subset U \times \sqrt{-1}\hat{\Xi}_1,$$

$$WF_*(u_2) \subset U \times \sqrt{-1}\hat{\Xi}_2,$$

where $\hat{\Xi}_i$ is the image of Ξ_i by the natural projection from $\mathbf{R}^n \setminus \{0\}$ to $S^{n-1}(i = 1, 2)$. Take a closed cone Ξ of $\mathbf{R}^n$ with $\Xi \cap ch(\Xi_1 \cup \Xi_2) = \emptyset$. Here $ch(\cdot)$ means the convex hull. Then

$$\Xi \cap \Xi_2 = \emptyset, (\Xi - \Xi_2) \cap \Xi_1 = \emptyset.$$

Let $\xi \in \Xi, \eta \in \Xi_2$. Since we have $|\xi - \eta| \geq c(|\xi| + \eta|)$ for some $C > 0$, we obtain for any $\epsilon > 0$ and for some $L > 0$ the estimate

$$\begin{aligned} |\hat{u}(\xi - \eta)\hat{v}(\eta)| &\leq C_\epsilon \exp\{M(\epsilon|\xi - \eta|)\} \\ &\quad \times C \exp M(L|\eta|) \\ &\leq C_\epsilon C \exp\{-M(c\epsilon|\xi| + c\epsilon|\eta|) \\ &\quad + M(L|\eta|)\}. \end{aligned}$$

Here, by Proposition 3.6. of H. Komatsu [3],

$$-M(p_1 + p_2) \leq -M\left(\frac{p_1}{H}\right) - M\left(\frac{p_2}{h}\right) + \log(AM_0)$$

holds for some $H \geq 1$. We obtain thereby

$$\begin{aligned} |\hat{u}(\xi - \eta)\hat{v}(\eta)| &\leq C_\epsilon C A \exp\{-M\left(\frac{c\epsilon}{H}|\xi|\right)\} \\ &\quad \times \exp\{-M\left(\frac{c\epsilon}{H}|\eta|\right) + M(L)|\eta|)\}. \end{aligned}$$

Hence we have

$$|\int_{\Xi_2} \hat{u}(\xi - \eta)\hat{v}(\eta)d\eta| \leq C_\epsilon \exp\{-M(\epsilon|\xi|)\}$$

for any $\epsilon > 0$ and $\xi \in \Xi$.

Next let $\xi \in \Xi, \eta \notin \Xi_2$. Take a small $\epsilon' > 0$ and suppose that

$$|\xi| \geq 1, |\eta| \leq \epsilon'|\xi|.$$

Then we have $\xi - \eta \notin \Xi_2$, and $|\xi - \eta| \geq (1-\epsilon')|\xi| \geq (1-2\epsilon')|\xi| + |\eta|$, which leads to the estimate

$$|\hat{u}(\xi-\eta)\hat{v}(\eta)| \leq C_{\epsilon,\epsilon''} \exp\{-M\Big(\frac{\epsilon(1-2\epsilon')}{H}|\xi|)\}$$
$$\times \exp\{-M\Big(\frac{\epsilon}{H}|\eta|\Big) - M(\epsilon^n|\eta|)\}$$

for any $\epsilon > 0$ and $\epsilon^n > 0$.

Finally let $\epsilon \in \Xi, \eta \notin \Xi_2$ and suppose that

$$|\eta| \geq \epsilon'|\xi|.$$

Then

$$|\eta| \geq (|\eta| + \epsilon'|\xi|)/2,$$

$$|\hat{u}(\xi-\eta)\hat{v}(\eta)| \leq C_{\epsilon''} \exp\{M(L|\xi-\eta|) - M\Big(\frac{\epsilon''}{2}|\eta| + \frac{\epsilon'\epsilon''}{2}|\xi|\Big)\}$$
$$\leq C_{\epsilon''} A \exp\{-M\Big(\frac{\epsilon'\epsilon''}{2H}|\xi|\Big)\} \exp\{M((1+\frac{1}{\epsilon'})L|\eta|) - M\Big(\frac{\epsilon''}{2H}|\eta|\Big)\}$$

hold for any $\epsilon'' > 0$. Hence we deduce

$$|\int_{\mathbf{R}^n\setminus\Xi_2} \hat{u}(\xi-\eta)\hat{v}(\eta)d\eta| \leq C_\epsilon\{-M(\epsilon|\xi|)\}$$

for any $\epsilon > 0$ and $\xi \in \Xi$. Thus, if we define the product of u and v by

$$u(x)v(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n\times\mathbf{R}^n} \hat{u}(\xi-\eta)\hat{v}(\eta)e^{\sqrt{-1}x\xi}d\eta d\xi,$$

we will obtain the desired result. It is easy to show $uv \in \mathcal{D}b^*(U)$.

Next we consider the integration.

Theorem 2. *Let $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^n$ be open sets, and let $f : U \times V \to V$ be the natural projection.*

If $f|$ suppu is a proper map for $u(t,x)$, an ultradistribution of class $*$, on $U \times V$, then the integration of $u(t,x)$ along the fiber

$$v(x) = \int_{f^{-1}(x)} u(t,x)dt$$

can be defined. Moreover wehave

$$WF_*(v) \subset \hat{w}(WF_*(u) \cap U \times \sqrt{-1}S^*V),$$

where $\tilde{\omega}$ denotes the natural projection from $U \times \sqrt{1-}S^*V$ to $\sqrt{-1}S^*V$.

Proof. We define the integration of ultradistributions as that of hyperfunctions. Refer to M. Sato et. al. [6]. We can obtain the estimate of WF_* immediately from Fourier trnasform of v.

Finally, we can easily consider the convolution of ultradistributions from the above argument.

Corollary 1. *Let $u(x)$ and $v(x)$ be ultradistributions of class $*$ on $\mathbf{R}^n$, either of which has compact support. Then the convolution $u(x) * v(x) \in \mathcal{D}b^*(\mathbf{R}^n)$ exists with the property:*

$$WF_*(u * v) \subset \{(x + y, \sqrt{-1}\xi\infty) | (x, \sqrt{-1}\xi\infty) \in WF_*(u),$$
$$(y, \sqrt{-1}\xi\infty) \in WF_*(v)\}.$$

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