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SOME COMMON FIXED POINT THEOREMS FOR MAPPINGS OF CONTRACTIVE TYPE

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Abstract

In this note we establish two common fixed point theorems for four mappings in a complete metric space and apply them to get two common fixed point theorems in probabilistic metric spaces. The last theorem improves a recent result of T. H. Chang [1].

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1. Two common fixed point theorems in metric spaces

Let S, T, I, J be four mappings of a metric space (X, d) into itself. We denote

$$m(Sx,Ty) = \max\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), \frac{1}{2}(d(Ix,Ty)+d(Jy,Sx))\}$$

We shall consider the mappings which satisfy the following condition: for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

(1) $$\varepsilon \le m(Sx,Ty) < \varepsilon + \delta \text{ implies } d(Sx,Ty) < \varepsilon.$$

Remark 1. Condition (1) implies

$$d(Sx,Ty) < m(Sx,Ty) \text{ if } m(Sx,Ty) > 0. \tag{2}$$

Indeed, if $m(Sx,Ty) > 0$ we put $\varepsilon = m(Sx,Ty)$. Then by (1) there exists a $\delta > 0$ such that (1) holds. Since $m(Sx,Ty) < \varepsilon + \delta$ we have $d(Sx,Ty) < \varepsilon = m(Sx,Ty)$, so (2) is proved. Moreover, $m(Sx,Ty) = 0$ implies $d(Sx,Ty) = 0$ because in this case we get $Sx = Ix = Jx = Ty$.

Remark 2. Condition (1) is equivalent to the following: for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$m(Sx,Ty) < \varepsilon + \delta \text{ implies } d(Sx,Ty) < \varepsilon. \tag{3}$$

It is clear that (3) implies (1), we shall prove the converse. Let (1) hold and $m(Sx,Ty) < \varepsilon + \delta$. If $m(Sx,Ty) \geq \varepsilon$, then by (1) we get $d(Sx,Ty) < \varepsilon$. If $0 < m(Sx,Ty) < \varepsilon$ then by Remark 1 we have $d(Sx,Ty) < m(Sx,Ty) < \varepsilon$. Finally, if $m(Sx,Ty) = 0$ we get $d(Sx,Ty) = 0 < \varepsilon$. Thus in any case we have $d(Sx,Ty) < \varepsilon$, so (3) holds.

Remark 3. In [3] Meir and Keeler considered the mappings called (ε, δ)-contractions which satisfy the following condition: for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \text{ implies } d(Tx,Ty) < \varepsilon.$$

By the same arguments as in Remark 2 it is easy to verify that the above condition is equivalent to the following

$$d(x,y) < \varepsilon + \delta \text{ implies } d(Tx,Ty) < \varepsilon$$

and now it is clear that Condition (1) is weaker than the Meir-Keeler's condition, even for the case when $S = T$ and I, J are the identity in X.

Before stating the main results of this note we need the following (see [2]).

Definition 1. *Two mappins S and T are said to be compatible if*

$$\lim_{n\to\infty} d(Ix_n, Sx_n) = 0 \text{ implies } \lim_{n\to\infty} d(SIx_n, ISx_n) = 0.$$

In particular, if S and T are compatible and $Ix = Sx$ then $SIx = ISx$.

Remark 4. This condition is weaker than the weak commutativity (see [7]), i. e.

$$d(SIx, ISx) \leq d(Ix, Sx) \text{ for all } x \in X,$$

which is obviously weaker than the commutativity of S and T.

Theorem 1. *Let (X, d) be a complete metric space, S, T, I, J four mappings of X into itself such that*

a) $S(X) \subset J(X)$, $T(X) \subset I(X)$,

b) S and I are compatible, so are T and J,

c) S and I (or T and J) are continuous,

d) Condition (1) holds.

Then, the four mappings have a unique common fixed point.

Before prooving the theorem we establish the following

Lemma 1. *Let S, T, I, J be as in Theorem 1. If there exists a point $z \in X$ such that $Sz = Iz$ or $Tz = Jz$, then the four mappings have a unique common fixed point.*

Proof. Let $Sz = Iz = v$. By compatibility of S and I we obtain $ISz = SIz$, i. e. $Sv = Iv$. Since $S(X) \subset J(X)$ there exists an $u \in X$ such that $Ju = Sz$. Then

$$\begin{aligned} & m(Sz, Tu) \\ = \; & \max\{d(Iz, Ju), d(Iz, Sz), d(Ju, Tu), \frac{1}{2}(d(Iz, Tu) + d(Ju, Sz))\} \\ = \; & d(Ju, Tu) = d(Sz, Tu). \end{aligned}$$

By Remark 2, $Tu = Ju = Sz = Iz = v$. Now we have

$$\begin{aligned} & m(Sv,Tu) \\ = \ & \max\{d(Iv,Ju), d(Iv,Sv), d(Ju,Tu), \frac{1}{2}(d(Iv,Tu)+d(Ju,Sv))\} \\ = \ & d(Sv,Tu) = d(Sv,v). \end{aligned}$$

By Remark 2 again, $Sv = v$. By compatibility of T and J we get $TJu = JTu$, i. e. $Tv = Jv$. Finally, we have

$$\begin{aligned} & m(Sv,Tv) \\ = \ & \max\{d(Iv,Jv), d(Iv,Sv), d(Jv,Tv), \frac{1}{2}(d(Iv,Tv)+d(Jv,Sv))\} \\ = \ & d(Sv,Tv). \end{aligned}$$

Thus we obtain $Sv = Tv = Iv = Jv = v$. Similarly for the case when $Tz = Jz$. Now suppose that $Sv' = Tv' = Iv' = Jv' = v'$, then $m(Sv,Tv') = d(Sv,Tv') = d(v,v')$, so $v = v'$ and the common fixed point v is unique. The lemma is prooved.

Remark that we do not use the continuity of the mappings in the proof.

In what follows we shall use the following notations: $\mathbf{N}$ stands for the set of all natural numbers. $\mathbf{R}$ is the set of all real numbers and $\mathbf{R}^+$ the set of all nonnegative numbers.

Proof of Theorem 1. Let $x_0 \in X$, since $S(X) \subset J(X)$ there exists an $x_1 \in X$ such that $Jx_1 = Sx_0$. Since $T(X) \subset I(X)$ there is an $x_2 \in X$ such that $Ix_2 = Tx_1$. Continuing this process we obtain a sequence $\{z_n\}$ such that

$$z_{2n} = Sx_{2n} = Jx_{2n+1},\ z_{2n+1} = Tx_{2n+1} = Ix_{2n+2},\ n = 0, 1, \ldots .$$

Denoting $c_n = d(z_n, z_{n+1})$ we have

$$\begin{aligned} m(z_{2n}, z_{2n+1}) &= m(Sx_{2n}, Tx_{2n+1}) \\ &= \max\{d(Ix_{2n}, Jx_{2n+1}), d(Ix_{2n}, Sx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2}(d(Ix_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, Sx_{2n}))\} \\ &= \max\{d(z_{2n-1}, z_{2n}), d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n+1}), \\ &\quad \frac{1}{2}(d(z_{2n-1}, z_{2n+1}) + d(z_{2n}, z_{2n}))\} \\ &= \max\{c_{2n-1}, c_{2n}\}. \end{aligned}$$

Similarly, we have $m(z_{2n}, z_{2n+1}) = m(Tx_{2n}, Sx_{2n+1}) = \max\{c_{2n}, c_{2n+1}\}$, so in general, we get $m(z_n, z_{n+1}) = \max\{c_{n-1}, c_n\}$ for all $n = 1, 2, \ldots$. We now prove that the sequence $\{z_n\}$ is convergent. We remark that if $m(z_k, z_{k+1}) = 0$ for some k, then $c_{k-1} = c_k = 0$ and hence $m(z_k, z_{k+1}) = \max\{c_k, c_{k+1}\} = c_{k+1}$. On the other hand we have $d(Sx_{2n}, Tx_{2n+1}) = d(z_{2n}, z_{2n+1}) = c_{2n}$, $d(Tx_{2n+1}, Sx_{2n+2}) = d(z_{2n+1}, z_{2n+2}) = c_{2n+1}$. If $c_{k+1} > 0$, by (2) we get $c_{k+1} < c_{k+1}$ a contradiction. So we have $c_{k+1} = 0$ and therefore $c_n = 0$ for all $n \geq k$, i. e. $z_n = z_k$ for all $n \geq k$ and $\{z_n\}$ is convergent. Thus we may assume that $m(z_n, z_{n+1}) > 0$ for each n. Then by (2) we have $c_n < \max\{c_{n-1}, c_n\}$, hence $c_{n-1} > c_n$ for each n, so $\{c_n\}$ monotonously converges to $\varepsilon \geq 0$. If $\varepsilon > 0$, then by (1) there exists a $\delta > 0$ such that $c_n < \varepsilon + \delta$ implies $c_{n+1} < \varepsilon$ contradicting the fact that $c_n \geq \varepsilon$ for every n. Thus $\{c_n\}$ converges to 0.

Following the method of Meir and Keeler we now prove that $\{z_n\}$ is a Cauchy sequence by contradiction: Suppose the contrary that there exists an $\varepsilon > 0$ such that for every $k \in \mathbf{N}$ thre are $n > m \geq k$ satisfying $d(z_m, z_n) \geq 2\varepsilon$. For this ε we choose a δ satisfying (1), then put $\alpha = \min\{\varepsilon, \delta\}$. Choose k such that $c_j < \frac{\alpha}{4}$ for all $j \geq k$ and take $n > m \geq k+1$ satisfying $d(z_m, z_n) \geq 2\varepsilon$. For each $j \in \{m, \ldots, n\}$ we have

$$d(z_m, z_j) \leq d(z_m, z_{j+1}) + c_j,\ d(z_m, z_{j+1}) \leq d(z_m, z_j) + c_j$$

This implies $|d(z_m, z_{j+1}) - d(z_m, z_j)| \leq c_j < \frac{\alpha}{4}$. Then, there exists an $l \in \{m, \ldots, n\}$ such that

$$\varepsilon + \frac{\alpha}{4} \leq d(z_m, z_l) \leq \varepsilon + \frac{\alpha}{2}. \tag{4}$$

Consider the case when m is odd and l is even. We have

$$\begin{aligned} m(z_m, z_l) &= m(Tx_m, Sx_l) \\ &= \max\{d(Ix_l, Jx_m), d(Ix_l, Sx_l), d(Jx_m, Tx_m), \\ &\quad \frac{1}{2}(d(Ix_l, Tx_m) + d(Jx_m, Sx_l))\} \\ &= \max\{d(z_{l-1}, z_{m-1}), d(z_{l-1}, z_l), d(z_{m-1}, z_m), \\ &\quad \frac{1}{2}(d(z_{l-1}, z_m) + d(z_l, z_{m-1}))\} \\ &< \varepsilon + \frac{\alpha}{2} + \frac{\alpha}{4} + \frac{\alpha}{4} = \varepsilon + \alpha \leq \varepsilon + \delta \end{aligned}$$

in view of (4) and by noting that $c_{m-1} < \frac{\alpha}{4}$, $c_{l-1} < \frac{\alpha}{4}$.

Since $m(Tx_m, Sx_l) < \varepsilon + \delta$, by (1) we get $d(Tx_m, Sx_l) = d(z_m, z_l) < \varepsilon$, acontradiction to (4).

Similarly for the case when m is even and l is odd.

For the case when both m and l are odd we have that $l - 1$ is even. Applying the result in the first case we get $m(z_m, z_{l-1}) < \varepsilon + \delta$. Then by (1) we obtain $d(z_m, z_{l-1}) < \varepsilon$. But, in view of (4) we have

$$d(z_m, z_{l-1}) \geq d(z_m, z_l) - c_{l-1} > \varepsilon + \frac{\alpha}{4} - \frac{\alpha}{4} = \varepsilon,$$

a contradiction. Similarly for the case when both m and l are even. Thus in any case we geet a contradiction and therefore $\{z_n\}$ is a Cauchy sequence which converges to a $z \in X$ since X is complete. By the construction of the sequence $\{z_n\}$ we have

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ix_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Jx_{2n+1} = z.$$

By the compatibility of S and I and of T and J we obtain

$$\lim_{n\to\infty} d(ISx_{2n}, SIx_{2n}) = 0, \ \lim_{n\to\infty} d(JTx_{2n+1}, TJx_{2n+1}) = 0.$$

Suppose that S and I are continuous, then we have $Iz = Sz$. Applying the lemma, the theorem follows. Similarly for the case when T and J are continuous. The proof is complete.

Theorem 1 generalizes a result in [8].

Remark 5. The following example shows that the continuity of S can not be ommited. Let $X = \{1, \frac{1}{2}, \ldots, \frac{1}{2^n}, \ldots, 0\}$, $I = J =$ identity, $S = T$ be defined as follows

$$S(\frac{1}{2^n}) = \frac{1}{2^{n+1}} \text{ for } n = 0, 1, \ldots, \ S(0) = 1.$$

If we define

$$\delta = \begin{cases} \varepsilon, \text{ if } 0 < \varepsilon < \frac{1}{2} \\ 1 - \varepsilon, \text{ if } \frac{1}{2} \leq \varepsilon < 1 \\ \text{arbitrary positive if } \varepsilon \geq 1, \end{cases}$$

then all conditions of Theorem 1 are satisfied except the continuity of S, and S has no fixed point.

Remark 6. The following example construcded ba N. H. Dien shows that one can not replace $m(Sx,Ty)$ by

$$M(Sx,Ty) = \max\{d(Ix,Jy), d(Ix,Sx), d(Jy,Ty), d(Ix,Ty) + d(Jy,Sx)\}.$$

Let

$$X = \{1,2,3,4\}$$
$$d(1,1) = d(2,2) = d(3,3) = d(4,4) = 0$$
$$d(1,2) = d(3,4) = 2,\ d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1$$
$$I = J = \text{identity}$$
$$S(1) = 3,\ S(2) = 4,\ S(3) = 4,\ S(4) = 3$$
$$T(1) = 2,\ T(2) = 1,\ T(3) = 2,\ T(4) = 1.$$

Then S, T, J, I satisfy $d(Sx,Ty) \le \frac{1}{2}M(Sx,Ty)$, so we can put $\delta = \varepsilon$ to get

$$\varepsilon \le M(Sx,Ty) < \varepsilon + \delta \text{ implies } d(Sx,Ty) < \varepsilon,$$

but S, T have not fixed point.

In Theorem 2 we shall replace Condition (1) by the following: there exists an upper-semicontinuous function $f : \mathbf{R}^+ \to \mathbf{R}^+$ satisfying $f(0) = 0$, $f(t) < t$ for $t > 0$ such that

(5) $$d(Sx,Ty) \le f(m(Sx,Ty)) \text{ for all } x, y \in X.$$

We remark that this condition is stronger than Condition (1). Indeed, suppose that (5) holds. For each $\varepsilon > 0$ we have $f(\varepsilon) < \varepsilon$. By the upper-semicontinuity of f, there exists a $\delta > 0$ such that $\varepsilon \le t < \varepsilon + \delta$ implies $f(t) < \varepsilon$. So $\varepsilon \le m(Sx,Ty) < \varepsilon + \delta$ implies $f(m(Sx,Ty)) < \varepsilon$ and hence by (5) $d(Sx,Ty) < \varepsilon$ i. e. Condition (1) holds.

The following result shows that with Condition (5) the continuoity of S or of I in Theorem 1 may be omitted.

Theorem 2. *Let X, S, T, I, J be as in Theorem 1 which satisfy the conditions a), b) of Theorem 1 and*

c) One of S, T, I, J is continuous,

d) Condition (5) holds.

Proof. We construct a Cauchy sequence $\{z_n\}$ as in the beginning of the ptoof of Theorem 1, where the continuoity of the mappings plays no role. We get also

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ix_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Jx_{2n+1} = z.$$

Suppose that I is continuous, then $I^2x_{2n} \to Iz$ and $ISx_{2n} \to Iz$. Since S and I are compatible, $SIx_{2n} \to Iz$ too. By (5) we have

$$\begin{aligned} & d(SIx_{2n}, Tx_{2n+1}) \\ \leq\ & f(\max\{d(I^2x_{2n}, Jx_{2n+1}), d(I^2x_{2n}, SIx_{2n}), d(Jx_{2n+1}, Tx_{2n+1}), \\ & \frac{1}{2}(d(I^2x_{2n}, Tx_{2n+1}) + d(Jx_{2n+1}, SIx_{2n}))\}). \end{aligned}$$

Letting $n \to \infty$ and using the upper semicontinuoity of f we get

$$\begin{aligned} & d(Iz, z) \\ \leq\ & f(\max\{d(Iz, z), d(Iz, z), d(z, z), \frac{1}{2}(d(Iz, z) + d(Iz, z))\}) \\ =\ & f(d(Iz, z)) < d(Iz, z) \text{ if } d(Iz, z) > 0. \end{aligned}$$

This shows that $Iz = z$. Now we have

$$\begin{aligned} & d(Sz, Tx_{2n+1}) \\ \leq\ & f(\max\{d(Iz, Jx_{2n+1}), d(Iz, Sz), d(Jx_{2n+1}, Tx_{2n+1}), \\ & \frac{1}{2}(d(Iz, Tx_{2n+1}) + d(Jx_{2n+1}, Sz))\}). \end{aligned}$$

Letting $n \to \infty$ and using the upper semicontinuoity of f we get

$$\begin{aligned} & d(Sz, z) \\ \leq\ & f(\max\{d(z, z), d(z, Sz), d(z, z), \frac{1}{2}(d(z, z) + d(z, Sz))\}) \\ =\ & f(d(Sz, z)) < d(Sz, z) \text{ if } d(Sz, z) > 0. \end{aligned}$$

This shows that $Sz = z$. Since $Iz = Sz = z$ we can apply the above lemma and Theorem 2 follows. Similarly for the case when J is continuous.

Now we suppose that S is continuous. Then $SJx_{2n+1} \to Sz$ and $SIx_2 \to Sz$. By (5) we obtain

$$\begin{aligned} & d(SJx_{2n+1}, Tx_{2n+1}) \\ \leq\ & f(\max\{d(IJx_{2n+1}, Jx_{2n+1}), d(IJx_{2n+1}, SJx_{2n+1}), d(Jx_{2n+1}, Tx_{2n+1}), \\ & \frac{1}{2}(d(IJx_{2n+1}, Tx_{2n+1}) + d(Jx_{2n+1}, SJx_{2n+1}))\}). \end{aligned}$$

Letting $n \to \infty$ and using the upper semicontinuoity of f we get

$$\begin{aligned} & d(Sz, z) \\ \leq\ & f(\max\{d(Sz, z), d(Sz, z), d(z, z), \frac{1}{2}(d(Sz, z) + d(z, Sz))\}) \\ =\ & f(d(Sz, z)) < d(Sz, z) \text{ if } d(Sz, z) > 0. \end{aligned}$$

Thus we get $Sz = z$. Since $S(X) \subset J(X)$ there exists an $u \in X$ such that $Ju = Sz = z$. Hence $TJu = Tz$, and by (5) we obtain

$$\begin{aligned} & d(SJx_{2n+1}, Tu) \\ \leq\ & f(\max\{d(IJx_{2n+1}, Ju), d(IJx_{2n+1}, SJx_{2n}), d(Ju, Tu), \\ & \frac{1}{2}(d(Ju, SJx_{2n+1}) + d(Jx_{2n+1}, Tu))\}). \end{aligned}$$

Letting $n \to \infty$ and using the upper semicontinuoity of f we get

$$\begin{aligned} & d(z, Tu) \\ \leq\ & f(\max\{d(z, z), d(z, z), d(z, Tu), \frac{1}{2}(d(z, z) + d(z, Tu))\}) \\ =\ & f(d(z, Tu)) < d(z, Tu) \text{ if } d(z, Tu) > 0. \end{aligned}$$

Thus $Tu = z = Ju$. Applying the above lemma, Theorem 2 follows. Similarly for the case when T is continuous. The proof is complete.

Remark that this theorem generalizes a result in [5]. Example 2 shows that in (5) $m(Sx, Ty)$ can not be replaced by $M(Sx, Ty)$.

2. Application to probabilistic metric spaces

Let X be arbitrary nonempty set, by $\mathcal{F}$ we denote a family of functions $F : X \times X \times \mathbf{R} \to [0, 1]$ satisfying the following conditions:

a) For every $x, y \in X$ the function $F_{x,y}(\cdot)$ is left-continuous, nondecreasing, $\inf_{t\in\mathbf{R}} F_{x,y}(t) = 0$, $\sup_{t\in\mathbf{R}} F_{x,y}(t) = 1$, i. e. $F_{x,y}$ is a distribution,

b) $F_{x,y}(0) = 0$ for every $x, y \in X$,

c) $F_{x,y}(t) = 1$ for all $t > 0$ if and inly if $x = y$,

d) $F_{x,y}(t) = F_{y,x}(t)$ for every $x, y \in X$, $t \in \mathbf{R}$,

e) $F_{x,y} \geq \min\{F_{x,z}, F_{z,y}\}$ for every $x, y \in X$, $t \in \mathbf{R}$.

Definition 2. *We shall call a set X with a family of distributions satisfying the above conditions a probabilistic metric space (or briefly, a PM-space) and denote it by $(X, \mathcal{F}, \min)$.*

In a PM-space one can define a topology as follows. A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$. A sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} F_{x_n,x_m}(t) = 1$ for all $t > 0$. A PM-space is said to be complete if every Cauchy sequence is convergent.

Remark that the condition c) anove ensures that the space $(X, \mathcal{F}, \min)$ is a separated (Hausdorf) topological space. The theory of PM-spaces was initiated by Menger in [4] and thoroughly investigated by Schweizer and Sklar in [6].

In the sequel we shall use the following notion. Let X, Λ be arbitrary sets, $\{d_\lambda;\ \lambda \in \Lambda\}$ a family of pseudo-metrics in X, i. e. for each $\lambda \in \Lambda$ d_λ is a function of $X \times X$ into $\mathbf{R}^+$ satisying for every $x, y, z \in X$

a) $d_\lambda(x, y) \geq 0$, $d_\lambda(x, x) = 0$,

a) $d_\lambda(x, y) = d_\lambda(y, x)$,

a) $d_\lambda(x, y) \geq d_\lambda(x, z) + d_\lambda(z, y)$.

In the space $(X, \{d_\lambda\})$ one can define a topology as follows. A sequence $\{x_n\}$ is said to be convergent to $x \in X$ if $d_\lambda(x_n, x) \to 0$ as $n \to \infty$ for each $\lambda \in \Lambda$. A sequence $\{x_n\}$ called a Cauchy sequence if $d_\lambda(x_n, x_m) \to 0$ as $n, m \to \infty$ for each $\lambda \in \Lambda$. The space $(X, \{d_\lambda\})$ is said to be complete if every Cauchy sequence is convergent. If we assume in addition that

$$d_\lambda(x, y) = 0 \text{ for all } \lambda \in \Lambda \text{ implies } x = y,$$

then the space $(X, \{d_\lambda\})$ becomes a separated topological space.

It is interesting to note that there is a close relation between metric spaces, PM-spaces and spaces with a family of pseudo-metrics as follows. Let a metric space (X, d) be given, then we can construct a corresponding PM-space $(X, \mathcal{F}, \min)$ by putting

$$F_{x,y} = \begin{cases} 0, \text{ if } t \le d(x,y) \\ 1, \text{ if } t > d(x,y) \end{cases}$$

On the other hand, if a PM-space $(X, \mathcal{F}, \min)$ is given, then one can construct a corresponding space with a family of pseudo-metrics $(X, \{d\}_\lambda)$ by putting $\Lambda = (0,1)$ and

$$d_\lambda(x,y) = \sup\{t : \ F_{x,y}(t) \le 1 - \lambda\}, \ \lambda \in (0,1).$$

It is not difficult to verify that the family $\{d_\lambda\}$ generates the same topology in $(X, \mathcal{F}, \min)$, in particular, $(X, \{d_\lambda\})$ is separated. By the left-continuity of $F_{x,y}$ it is clear that

$$(6) \qquad F_{x,y}(d_\lambda(x,y)) \le 1 - \lambda \text{ for every } x, y \in X, \ \lambda \in (0,1),$$

and therefore

$$(7) \qquad F_{x,y}(d_\lambda(x,y)) > 1 - \lambda \text{ if and only if } t > d(x,y).$$

Now we can extend Theorems 1 and 2 to spaces with a family of pseudo-metrics.

Proposition 1. *Let $(X, \{d_\lambda\})$ be a complete separated space with a family of pseudo-metrics, S, T, I, J four mappings in X satisfying for each $\lambda \in \Lambda$ the conditions of either Theorem 1 or Theorem 2 with d replaced by d_λ. Then, the four mappings have a unique common fixed point.*

The proof of this proposition can be omitted because one must only repeat the same proof of Theorems 1 and 2 with d replaced by d_λ and noting that the constructed sequence $\{z_n\}$ is independent of λ and that the space $(X, \{d_\lambda\})$ is assumed separated.

Definition 3. *Two mappings S and I of a PM-space $(X, \mathcal{F}, \min)$ into itself are said to be probabilistic compatible if*

$$\lim_{n\to\infty} F_{SIx_n, ISx_n}(t) = 1 \textit{ for every } t > 0$$

whenever we have

$$\lim_{n\to\infty} F_{Sx_n,Ix_n}(t) = 1 \text{ for every } t > 0.$$

Note that this definition was introduced by Chang in [1], where the word *probabilistic* was ommited. We now can state two results in PM-spaces analogous to Theorem 1 and 2.

Theorem 3. *Let* $(X, \mathcal{F}, \min)$ *be a complete PM-space,* S, T, I, J *four mappings in* X *such that*

a) $S(X) \subset J(X)$, $T(X) \subset I(X)$,

b) S *and* I *are probabilistic compatible, and so are* T *and* J,

c) S *and* I *(or* T *and* J*) are continuous,*

d) For each $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for every* $x, y \in X$

$$\begin{aligned}(8)\ F_{Sx,Ty}(\varepsilon) \geq\ & \min\{F_{Ix,Jy}(\varepsilon+\delta), F_{Ix,Sx}(\varepsilon+\delta), F_{Jy,Ty}(\varepsilon+\delta),\\ & \max(F_{Ix,Ty}(\varepsilon+\delta), F_{Jy,Sx}(\varepsilon+\delta))\}\end{aligned}$$

Then S, T, I, J *have a unique common fixed point.*

Proof. We construct the corresponding space with a family od pseudometrics by putting $d_\lambda(x,y) = \sup\{t :\ F_{x,y}(t) \leq 1-\lambda,\ \lambda \in (0,1)\}$. It is clear that the conditions a), b), c) of Theorem 3 ensure the corresponding conditions of Theorem 1. So, it remains to verify the condition (1) with d replaced d_λ. Letting $\varepsilon > 0$ be given, we choose a $\delta > 0$ such that (8) holds. If $m_\lambda(Sx,Ty) < \varepsilon + \delta$ then by (7)

$$F_{Ix,Jy}(\varepsilon+\delta) > 1-\lambda,\ F_{Ix,Sx}(\varepsilon+\delta) > 1-\lambda,\ F_{Jy,Ty}(\varepsilon+\delta) > 1-\lambda.$$

To show that $\max\{F_{Ix,Ty}(\varepsilon+\delta), F_{Jy,Sx}(\varepsilon+\delta)\} > 1-\lambda$ we suppose the contrary that both of $F_{Ix,Ty}(\varepsilon+\delta)$ and $F_{Jy,Sx}(\varepsilon+\delta) \leq 1-\lambda$ which imply by (7) both of $d_\lambda(Ix,Ty)$ and $d_\lambda(Jy,Sx) \geq \varepsilon+\delta$, so we have

$$\frac{1}{2}(d_\lambda(Ix,Ty) + d_\lambda(Jy,Sx)) \geq \varepsilon + \delta$$

contradicting the assumption that $m_\lambda(Sx,Ty) < \varepsilon + \delta$. Thus,

$$\min\{F_{Ix,Jy}(\varepsilon+\delta), F_{Ix,Sx}(\varepsilon+\delta), F_{Jy,Ty}(\varepsilon+\delta),$$
$$\max(F_{Ix,Ty}(\varepsilon+\delta), F_{Jy,Sx}(\varepsilon+\delta))\} > 1-\lambda.$$

By (8) we get $F_{Sx,Jy}(\varepsilon) > 1-\lambda$. Applying (7) we obtain $d_\lambda(Sx,Ty) < \varepsilon$, so (1) holds. By the above Proposition, S, T, I, J have a unique common fixed point.

By the same method and using Theorem 2 we can prove the following

Theorem 4. *Let $(X,\mathcal{F},\min)$ and S, T, I, J be as in Theorem 3 which satisfy the conditions a), b) of Theorem 3 and*

c) One of the mappings is continuous,

d) There exists an upper-semicontinuous function $g : \mathbf{R}^+ \to \mathbf{R}^+$ such that $g(0) = 0$, $g(t) < t$ for $t > 0$ and

$$F_{Sx,Ty}(g(t)) \geq \min\{F_{Ix,Jy}(t), F_{Ix,Sx}(t), F_{Jy,Ty}(t), \max(F_{Ix,Ty}(t), F_{Jy,Sx}(t))\} \tag{9}$$

for every $x, y \in X$ and $t \in \mathbf{R}$.

Then the four mappings have a unique common fixed point.

Proof. By Proposition 1 in [1] there exists a continuous and strictly increasing (hence invertible) function $f : \mathbf{R}^+ \to \mathbf{R}^+$ such that $g(t) \leq f(t) < t$ for every $t > 0$. Since $F_{Sx,Ty}(\cdot)$ is nondecreasing, from (9) we get

$$F_{Sx,Ty}(f(t)) \geq \min\{F_{Ix,Jy}(t), F_{Ix,Sx}(t), F_{Jy,Ty}(t), \max(F_{Ix,Ty}(t), F_{Jy,Sx}(t))\} \tag{10}$$

for every $x, y \in X$ and $t \in \mathbf{R}$. Putting $d_\lambda(x,y) = \sup\{t :\ F_{x,y}(t) \leq 1-\lambda\}$, $\lambda \in (0,1)$, we now show that the condition (5) holds with d replaced by d_λ. Suppose the contrary that there exist x, y, λ such that

$$d_\lambda(Sx,Ty) > f(m_\lambda(Sx,Ty)),$$

from this

$$f^{-1}(d_\lambda(Sx,Ty)) > m_\lambda(Sx,Ty)$$

by using the strict monotony of f. Denoting $t = f^{-1}(d_\lambda(Sx,Ty))$ we get $m_\lambda(Sx,Ty) < t$ which implies

$$\min\{F_{Ix,Jy}(t), F_{Ix,Sx}(t), F_{Jy,Ty}(t), \max(F_{Ix,Ty}(t), F_{Jy,Sx}(t))\} > 1-\lambda$$

as in the proof of Theorem 3. By (10) we get

$$F_{Sx,Ty}(f(t)) = F_{Sx,Ty}(d_\lambda(Sx,Ty)) > 1-\lambda,$$

a contradiction to (6). Thus the condition (5) holds and the proof is complete by applying the above Proposition.

Theorem 4 improves a result in [1] where $\max\{F_{Ix,Ty}(t), F_{Jy,Sx}(t)\}$ is replaced by $F_{Ix,Ty}(t) + F_{Jy,Sx}(t)$ in Condition (9). The Remark 6 shows that the symbol max in (9) can not be ommitted.

References

[1] Chang, T. H., Common fixed point theorems inMenger spaces, Bull. Inst. Math. Acad. Sinica 22,1 (1994), 17-29.

[2] Jungck, G., Compatible mappings and common fixed points, Internat. J. Math. Sci. 4 (1986), 771-779.

[3] Meir, A., Keeler, E., A theorem on contractive mappings, J. Math. Anal. Appl. 28 (1969), 326-329.

[4] Menger, K., Statistical metrics, Proc. Nat. Acad. Sci. USA 28 (1942), 535-537.

[5] Rhoades, B. E., Park, S., Moon, R. B., On generalizations of the Meir-Keeler type contraction maps, J. Math. Anal. Appl. 146 (1990), 482-494.

[6] Schwieser, B., Sklar, A., Statistical metric spaces, Pacif. J. Math. 10 (1960), 313-334.

[7] Sessa, S., Fisher, B., Common fixed points of weakly commuting mappings, Bull. Polish. Acad. Sci. Math. 25 (1987), 341-349.

[8] Tan, D. H., Munh, N. A., Some fixed point theorems for mappings of contractive type, Acta Math. Nietnam 3,1 (1978), 24-42.

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