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SUFFICIENT CONDITIONS FOR UNIVALENCE OF LARGE CLASS OF FUNCTIONS

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Abstract

We present a sufficient condition for the analyticity and univalence of the functions of the form

$$F(z) = (\alpha \int_0^z \frac{f^\alpha(u)}{u} du)^{1/\alpha}$$

using the method of subordination chains. They appear in the integral representation of α-convex functions, a concept introduced by P. T. Mocanu.

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1. Intoduction

We denote by U_r the disk of the z-plane, $U_r = \{z \in \mathbf{C} : |z| < r\}$ where $r \in (0,1]$, $U_1 = U$, $U^* = U \setminus \{0\}$ and $I = [0, \infty)$.

Let A be the class of functions f which are analytic in U with $f(0) = 0$ and $f'(0) = 1$.

Theorem A. ([1]) *Let $f \in A$ and let α be a complex number. If the following conditions are satisfied*

(1) $$|\alpha - 1| < 1$$

(2) $$|\alpha(1-|z|^2)(\frac{zf'(z)}{f(z)} - 1) + \alpha - 1| \leq 1, \ (\forall) z \in U^*,$$

then the function

(3) $$g_\alpha(z) = (\alpha \int_0^z \frac{f^\alpha}{u} du)^{1/\alpha}$$

is analytic and univalent in U.

2. Preliminaries

Definition 1. *A function $L : U \times I \to \mathbf{C}$ is called Loewner's chain if:*

$$L(z,t) = e^t z + a_2(t)z^2 + \ldots, \ |z| < 1$$

is analytic and univalent in U for each $t \in I$ and if

$$L(z,s)\rho L(z,t), \ 0 \leq s < t < \infty,$$

where by ρ we denote the relation of subordination.

Theorem B. ([3]) *Let r be a real number $r \in (0,1]$. Let $L(z,t) = a_1 z + a_2 z^2 + \ldots$, $a_1 \neq 0$, be analytic in U_r for all $t \in I$, locally absolutely continuous in I and locally uniform with respect to U_r. For almost all $t \in I$ suppose*

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{L(z,t)}{\partial t}, \ (\forall) z \in U_r,$$

where $p(z,t)$ is analytic in U and satisfies $\mathrm{Re}\, p(z,t) > 0$, $z \in U$, $t \in I$.

If $|a_1(t)| \to \infty$ for $t \to \infty$ and $\{L(z,t)/a_1(t)\}$ forms a normal family in U_r, then, for each $t \in I$, $L(z,t)$ there is an analytic and univalent extension to whole disc U.

3. Main results

Theorem 1. *Let $f \in A$ and let α and c be complex numbers $|\alpha - a| < 1$, $|c| < 1$. If there exists an analytic function g, $g \in A$, such that*

$$(4) \qquad |\alpha(1+c)g'(z) - 1| \leq 1, \ (\forall) z \in U$$

$$(5) \qquad |(\alpha(1+c)g'(z) - 1)|z|^2 + (1-|z|^2)(\frac{\alpha z f'(z)}{f(z)} - \frac{zg''(z)}{g'(z)} - 1)| \leq 1$$

for all $z \in U$, then the function F,

$$(6) \qquad F(z) = (\alpha \int_0^z \frac{f^\alpha(u)}{u} du)^{1/\alpha}$$

is analytic and univalent in U.

Proof. Let us prove that there exists a real number r, $r \in (0,1]$, such that the function $L : U_r \times I \to \mathbf{C}$, defined formally by

$$(7) \qquad L(z,t) = (\alpha \int_0^{e^{-t}z} \frac{f^\alpha(u)}{u} du + \frac{e^{2t}-1}{1+c} \cdot \frac{f^\alpha(e^{-t}z)}{g'(e^{-t}z)})^{1/\alpha}.$$

is analytic in U_r for all $t \in I$.

Because $f \in A$, it results that the function h,

$$(8) \qquad h(z) = \frac{f(z)}{z} = 1 + b_1 z + \ldots + b_n z^n + \ldots$$

is analytic in U and $h(0) = 1$. Then, there exists r_1, $0 < r_1 \leq 1$, such that $h(z)$ does not vanish in U_{r_1}. In this case we denote by $h_1(z)$ the uniform branch of $(h(z))^\alpha$ which satisfies $h_1(0) = 1$ and is analytic in U_{r_1}

$$(9) \qquad h_1(z) = 1 + a_1 z + \ldots + a_n z^n + \ldots, \ (\forall) z \in U_{r_1}.$$

Let us denote by

$$(10) \qquad h_2(z,t) = \alpha \int_0^{e^{-t}z} h_1(u) u^{\alpha-1} du.$$

We obtain

$$h_2(z,t) = (e^{-t}z)^\alpha + \frac{a_1 \alpha}{1+\alpha}(e^{-t}z)^{\alpha+1} + \ldots$$

and we observe that

$$h_2(z,t) = (e^{-t}z)^{\alpha} h_3(z,t). \tag{11}$$

where

$$h_3 = 1 + \sum_{n=1}^{\infty} \frac{a_n \alpha}{n+\alpha} e^{-nt} z^n \tag{12}$$

The function h_3 is analytic in U_{r_1} for all $t \in I$, since

$$\overline{\lim_{n\to 0}} \left| \frac{a_n \alpha}{n+\alpha} e^{-nt} \right|^{1/2} = e^{-t} \overline{\lim_{n\to 0}} |a_n|^{1/2}.$$

From (4), which implies $g'(z) \neq 0$ for all $z \in U$, and from the analyticity of g in U, it follows that the function h_4 is also analytic in U_{r_1},

$$h_4(z,t) = h_3(z,t) + \frac{e^{2t} - 1}{1+c} \frac{h_1(e^{-t}z)}{g'(e^{-t}z)}. \tag{13}$$

From the hypothesis we have $|c| > 1$ and then $h_4(0,t) = (e^{2t} + c)/(1+c)$ does not vanish in U_{r_1} for all $t \in I$. It results there is a disc U_r, $0 < r < r_1$ in which $h_4(z,t) \neq 0$ for all $t \in I$. Then we can chose an analytic branch of $(h_4(z,t))^{1/\alpha}$, denoted by $h(z,t)$, which is equal to $((e^{2t} + c)/(1+c))^{1/\alpha}$ at the origin (for $((e^{2t} + c)/(1+c))^{1/\alpha}$ we fixed the determination equal to 1 for $t = 0$).

From (8)-(13) we have that the relation (7) may be written as

$$L(z,t) = e^{-t} z \cdot h(z,t) \tag{14}$$

and hence we obtain that the function $L(z,t)$ is analytic in U_r,

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots, (\forall) z \subset U_r,$$

where

$$a_1(t) = e^{\frac{2-\alpha}{\alpha}t} \cdot \left(\frac{1 + ce^{-2t}}{1+c}\right)^{1/\alpha}.$$

Because $|\alpha - 1| < 1$ is equivalent to $\mathrm{Re}\,\frac{2-\alpha}{\alpha} > 0$, we have $\lim_{t\to\infty} |a_1(t)| = \infty$. Also, because $|c| < 1$ we have $a_1(t) \neq 0$ for all $t \in I$.

$L(z,t)$ is an analytic function in U_r for all $t \geq 0$ and then we have that there exists a number r_2, $0 < r_2 < r$ and a constant $K = K(r_2)$, such that

$$\left| \frac{L(z,t)}{a_1(t)} \right| < K, \ (\forall) z \in U_{r_2},\ t \geq 0.$$

Then, by Montel's theorem, it results that $\{L(z,t)/a_1(t)\}$ is a normal family in U_{r_2}.

From (14) we have

$$\frac{\partial L(z,t)}{\partial t} = e^{-t}z \cdot (-h(z,t) + \frac{\partial h(z,t)}{\partial t}). \tag{15}$$

It is clear that $\frac{\partial h(z,t)}{\partial t}$ is an analytic function in U_{r_2} and then $\frac{\partial L(z,t)}{\partial t}$ is also an analytic function. Then, for all fixed numbers T and r_3, $T > 0$, $0 < r_3 < r_2$, there exists a constant K_1, $K_1 > 0$ (which depends on T and r_3), such that

$$\left|\frac{\partial L(z,t)}{\partial t}\right| < K_1,\ (\forall) z \in U_{r_3},\ t \in [0,T].$$

Therefore, the function $L(z,t)$ is locally absolutely continuous in $[0,\infty)$, locally uniform with respect to U_{r_3}.

Since $\frac{\partial L(z,t)}{\partial t}$ is analytic in U_{r_3}, it results from (15) that there is a number r_0, $0 < r_0 < r_3$, such that $\frac{1}{z}\frac{\partial L(z,t)}{\partial t} \neq 0$, $(\forall) z \in U_{r_0}$, and then the function

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} / \frac{\partial L(z,t)}{\partial t}$$

is analytic in U_{r_0} for all $t \geq 0$.

In order to prove that the function $p(z,t)$ has an analytic extension with positive real part in U, for all $t \geq 0$, it is sufficient to prove that the function $w(z,t)$ defined in U_{r_0} by

$$w(z,t) = \frac{p(z,t) - 1}{p(z,t) + 1} \tag{16}$$

can be continued analytically in U and $|w(z,t)| < 1$ for all $z \in U$ and $t \geq 0$.

After computation we obtain

$$\begin{aligned} w(z,t) &= \alpha(1+c)e^{-2t}g'(e^{-t}z) - 1 \\ &+ (1 - e^{-2t})e^{-t}z(\alpha\frac{f'(e^{-t}z)}{f(e^{-t}z)} - \frac{g''(e^{-t}z)}{g'(e^{-t}z)}). \end{aligned}$$

From (4) and (5) we obtain that $g'(z) \neq 0$ and $f(z) \neq 0$ for all $z \in U$ and then the function $w(z,t)$ is analytic in the unit disc U.

For $t = 0$, in view of (4) we have

$$(17) \qquad |w(z,0)| = |\alpha(1+c)g'(z) - 1| < 1, \ (\forall) z \in U.$$

From the hypothesis $|\alpha - 1| < 1$ and from (4) we have that

$$|\alpha(1+c)g'(0) - 1| = |\alpha(1+c) - 1| < 1$$

and for $z = 0$ and $t > 0$, we have

$$(18) \qquad \begin{aligned} |w(0,t)| &= |\alpha(1+c)e^{-2t} - 1 + \alpha(1-e^{-2t})| \\ &\leq |\alpha(1+c) - 1|e^{-2t} + |\alpha - 1|(1-e^{-2t}) < 1. \end{aligned}$$

Let now be a fixed number t, $t > 0$, $z \in U$, $z \neq 0$. In this case the function $w(z,t)$ is analytic in U because $|e^{-t}z| \leq e^{-t} < 1$ for all $z \in U$. Using the maximum principle, for all $z \in U$ and $t > 0$ we have

$$(19) \qquad |w(z,t)| < \max_{|\xi|=1} |w(\xi,t)| = |w(e^{i\theta},t)|,$$

where $\theta = \theta(t)$ is a real number.

Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and from (16) we obtain

$$\begin{aligned} |w(e^{i\theta},t)| &= |\alpha(1+c)g'(u)|u|^2 - 1 + (1-|u|^2)(\alpha\frac{uf'(u)}{f(u)} - \frac{ug''(u)}{g'(u)}) \\ &= |(\alpha(1+c)g'(u) - 1)|u|^2 + (1-|u|^2)(\alpha\frac{uf'(u)}{f(u)} - \frac{ug''(u)}{g'(u)} - 1) \end{aligned}$$

Because $u \in U$, the relation (5) implies $|w(e^{i\theta},t) \leq 1$ and from (17), (18) and (19) we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \geq 0$.

From Theorem B, we have that the function $L(z,t)$ has an analytic and univalent extension to the whole disc U, for each $t \in I$. For $t = 0$ we conclude that the function

$$L(z,0) = (\alpha \int_0^z \frac{f^\alpha(u)}{u} du)^{1/\alpha} \equiv F(z)$$

is analytic and univalent in U.

Remark. For $g(z) = z$, from Theorem 1 we get the following

Theorem 2. *Let $f \in A$ and let α and c be complex numbers, $|\alpha - 1| < 1$, $|c| < 1$ and $|\alpha(1+c) - 1| < 1$. If*

$$|(\alpha(1+c)-1)|z|^2 + (1-|z|^2)(\frac{\alpha z f'(z)}{f(z)} - 1)| \leq 1 \tag{20}$$

for all $z \in U^*$, then the function F, defined by (6) is analytic and univalent in U.

Corollary 1. *Let $f \in A$, $\alpha, c \in \mathbf{C}$, $|\alpha-1| < 1$, $|c| < 1$ and $|\alpha(1+c)-1| < 1$. If*

$$|\frac{\alpha z f'(z)}{f(z)} - 1| < 1, \ (\forall) z \in U,$$

then, the function F defined by (6) is analytic and univalent in U.

Remark. In the particular case $c = 0$, Theorem 2 becomes Theorem A. Indeed, from (20) we obtain

$$\begin{aligned} &|(\alpha-1)|z|^2 + (1-|z|^2)(\frac{\alpha z f'(z)}{f(z)} - 1)| \\ = \ &|\alpha(1-|z|^2)(\frac{z f'(z)}{f(z)} - 1) + \alpha - 1| \leq 1, \end{aligned}$$

which is just the relation (2).

Theorem 3. *Let $f \in A$ and let α and c be complex numbers, $|\alpha - 1| < 1$, $|c| < 1$. If there exists an analytic function p with positive real part in U, $p(0) = (2 - \alpha(1+c))/(\alpha(1+c))$, such that*

$$|\frac{p(z)-1}{p(z)+1}|z|^2 - (1-|z|^2)(\frac{\alpha z f'(z)}{f(z)} + \frac{z p'(z)}{p(z)+1} - 1)| \leq 1, \tag{21}$$

for all $z \in U^*$, then the function F defined by (6) is analytic and univalent in U.

Proof. Let p be an analytic function in U,

$$p(0) = (2-\alpha(1+c))/(\alpha(1+c)), \ \mathrm{Re}\, p(z) > 0, \ (\forall) z \in U.$$

If in Theorem 1 we consider the function g, $g \in A$,

$$g'(z) = \frac{2}{\alpha(1+c)(1+p(z))},$$

then by the inequality (5) becomes (21) and the inequality (4) is true because $\operatorname{Re} p(z) > 0$, $(\forall) z \in U$.

Remark. The function F defined by (6) is just Mocanu's operator which appears in the study of alpha-convex functions ([2]).

References

[1] Blezu, D., Drăghici, E., Ovesea, H., Pascu, N. N., Sufficient conditions for univalence of a large class of functions, Studia Univ. Babeş-Bolyai, Mathematica, XXXV, 4 (1990), 41-47.

[2] Mocanu, P. T., Une propriété de convexité généralisée dans la theorie de la representation conforme, Mathematica (Cluj), 11(34), 1 (1969), 127-133.

[3] Pommerenke, Ch., Uber die Subordination analytischer Functionen, J. Reine Angew. Math. 218 (1965), 157-173.

[4] Pommerenke, Ch., Univalent Functions, Vanderhoect und Ruprecht, Götingen, 1975.

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