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GAUSSIAN COLOMBEAU GENERALIZED RANDOM PROCESSES

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Abstract

We give the definition of a Gaussian Colombeau generalized process and necessary and sufficient conditions for existence of a Gaussian Colombeau generalized process in terms of its correlation function which is a Colombeau generalized function. We consider the Wiener process in Colombeau's sense.

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1. Introduction

Generalized random processes (g.r.p.) with paths in Colombeau space of new generalized functions $\mathcal{G}$ were introduced and analyzed in [5] and [1] in connection with stochastic differential equations. There are two different ways of introducing these g.r.p.

We can consider Colombeau g.r.p. as a class of functions which can be represented by the mapping $(\omega, \varphi, x) \mapsto R(\omega, \varphi, x)$, ($\omega$ belongs to the probability space Ω, φ is in a certain functions space and $x \in T$, T is an interval of the real line $\mathbf{R}$) such that R is measurable as function of ω, for every fixed φ and x, and for almost every ω $R(\omega,\cdot,\cdot)$ is in Colombeau space $\mathcal{E}_M$. Or, it may be treated as a C^∞ mapping from T to $L^p(\Omega)$ with appropriate estimates:

By following an idea of [1] we define a subclass of the space of C g.r.p. with paths in $\mathcal{G}$ which enables us to introduce and analyse the Gaussian Colombeau g.r.p. Also, we present the Wiener process in this contex.

2. Basic notions

Let T be an open subset of $\mathbf{R}$ and $C_0^\infty(T)$ the space of complex valued functions defined on $\mathbf{R}$ with compact supports contained in T. Denote

$$\mathcal{A}_0(\mathbf{R}) = \{\varphi \in C_0^\infty(\mathbf{R});\ \textstyle\int \varphi(x)dx = 1\},$$

and for $q \in \mathbf{N}_0$,

$$\mathcal{A}_q(\mathbf{R}) = \{\varphi \in \mathcal{A}_0;\ \textstyle\int x^j\varphi(x)dx = 0,\ 0 < j < q\}.$$

Also,

$$A_q(\mathbf{R}^n) = \{\phi \in C_0^\infty(\mathbf{R}^n);\ \phi(x_1,...x_n) = \textstyle\prod_{i=1}^n \varphi(x_i),\ \varphi \in \mathcal{A}_0(\mathbf{R})\}.$$

Put $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})$, $\phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\phi(\frac{x}{\varepsilon})$, $\check{\varphi}(x) = \varphi(-x)$, $x \in \mathbf{R}$.

The basic space $\mathcal{E}(T)$ consists of all the functions $R : \mathcal{A}_0(\mathbf{R}) \to C^\infty(T)$. It is an algebra with multiplication. More important is the subalgebra of moderate elements $\mathcal{E}_M(T)$:

$$\begin{aligned}\mathcal{E}_M(T) = \ & \{R(\varphi,x) \in \mathcal{E}(T);\ \forall K \subset\subset T,\\ & \forall\alpha \in \mathbf{N}_0,\ \exists N \in \mathbf{N}_0,\ \forall\varphi \in \mathcal{A}_N,\ \exists\eta,\ \exists C > 0,\\ & sup\{|\partial^\alpha R(\varphi_\varepsilon,x)|,\ x \in K\} \le C\varepsilon^{-N},\ \ 0 < \varepsilon < \eta\}\end{aligned}$$

Denote by Γ the set of sequences $\{a_q\}$ with positive elements which strictly increase to infinity. Then, the set of null elements $\mathcal{N}(T)$ in $\mathcal{E}(T)$ is defined as follows:

$$\begin{aligned}\mathcal{N}(T) = \ & \{R \in \mathcal{E}(T);\ \forall K \subset\subset T,\\ & \forall\alpha \in \mathbf{N}_0,\ \exists N \in \mathbf{N}_0,\ \exists\{a_q\} \in \Gamma,\ \forall q \ge N,\ \forall\varphi \in \mathcal{A}_q,\ \exists\eta,\ \exists C > 0,\\ & sup\{|\partial^\alpha R(\varphi_\varepsilon,x)|,\ x \in K\} \le C\varepsilon^{a_q - N},\ \ 0 < \varepsilon < \eta\}.\end{aligned}$$

The space of generalized functions on T, $\mathcal{G}(T)$ is defined by

$$\mathcal{G}(T) = \mathcal{E}_M(T)/\mathcal{N}(T).$$

The multiplication and derivation in $\mathcal{G}(T)$ are given by

$$RG = [R(\varphi,\cdot)G(\varphi,\cdot)],\quad \partial^\alpha R = [\partial^\alpha R(\varphi,\cdot)],$$

where by $[R]$ is denoted the class of equivalence in $\mathcal{G}(T)$ with the representative $R(\varphi, \cdot)$.

3. Colombeau generalized random processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space and T be an open interval of real line. Denote by $\mathcal{E}(\Omega, T)$ the space of functions $R : (\Omega, \mathcal{A}_0, T) \to \mathbf{C}$ such that for every $\varphi \in \mathcal{A}_0$, $(\omega, x) \mapsto R(\omega, \varphi, x)$ is a C^∞ function in x and $\{R(\cdot, \varphi, x);\ (\varphi, x) \in \mathcal{A}_0 \times T\}$ is a compatible family of random variables. Let

$$\begin{aligned}\mathcal{E}_M(\Omega,T) = \ & \{R(\omega,\varphi,x) \in \mathcal{E}(\Omega,T);\ \text{a.e. } \omega\ \forall K \subset\subset T,\\ & \forall\alpha \in \mathbf{N}_0,\ \exists N \in \mathbf{N}_0,\ \forall\varphi \in \mathcal{A}_N,\ \exists\eta,\ \exists C > 0,\\ (*) \qquad & sup\{|\partial^\alpha R(\omega,\varphi_\varepsilon,x)|,\ x \in K\} \le C\varepsilon^{-N},\ \ 0 < \varepsilon < \eta\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}(\Omega,T) = \ & \{R \in \mathcal{E}(\Omega,T);\ \text{a.e. } \omega,\ \forall K \subset\subset T,\ \forall\alpha \in \mathbf{N}_0,\\ & \exists N \in \mathbf{N}_0,\ \exists\{a_q\} \in \Gamma,\ \forall q \ge N,\ \forall\varphi \in \mathcal{A}_q,\ \exists\eta,\ \exists C > 0,\\ & sup\{|\partial^\alpha R(\omega,\varphi_\varepsilon,x)|,\ x \in K\} \le C\varepsilon^{a_q - N},\ \ 0 < \varepsilon < \eta\}.\end{aligned}$$

Then we define

$$\mathcal{G}(\Omega,T) = \mathcal{E}_M(\Omega,T)\ /\ \mathcal{N}(\Omega,T).$$

The Colombeau theory of generalized functions can be adapted to the vector valued case $L^p(\Omega)$ (see [5] and [1]).

We define $\mathcal{E}(T, L^p(\Omega))$, $p \ge 1$ as a space of all functions

$$\mathcal{A}_0 \times T \ni (\varphi, x) \mapsto R(\cdot, \varphi, x) \in L^p(\Omega)$$

which are C^∞ mapping for each fixed $\varphi \in \mathcal{A}_0$. The space of moderate elements in $\mathcal{E}(T, L^p(\Omega))$ is denoted by $\mathcal{E}_M(T, L^p(\Omega))$:

$$\begin{aligned}\mathcal{E}_M(T,L^p(\Omega)) = \ & \{R(\omega,\varphi,x) \in \mathcal{E}(T,L^p(\Omega)); \forall K \subset\subset T,\ \forall\alpha \in \mathbf{N}_0,\\ & \exists N \in \mathbf{N}_0,\ \forall\varphi \in \mathcal{A}_N,\ \exists\eta, C > 0,\\ (**) \qquad & ||\partial^\alpha R(\varphi_\varepsilon,x)||_p \le C\varepsilon^{-N},\ \ x \in K,\ 0 < \varepsilon < \eta.\}\end{aligned}$$

The set of null elements is:

$$\begin{aligned}\mathcal{N}(T,L^p(\Omega)) = \ & \{R(\omega,\varphi,x) \in \mathcal{E}(T,L^p(\Omega)); \forall K \subset\subset T,\ \forall\alpha \in \mathbf{N}_0,\\ & \exists N \in \mathbf{N}_0,\ \exists\{a_q\} \in \Gamma,\ \forall q \ge N,\ \forall\varphi \in \mathcal{A}_q,\ \exists\eta, C > 0,\\ & ||\partial^\alpha R(\varphi_\varepsilon,x)||_p \le C\varepsilon^{a_q - N},\ \ x \in K,\ 0 < \varepsilon < \eta.\}\end{aligned}$$

The space of Colombeau generalized random processes (C g.r.p.) is then defined as:

$$\mathcal{G}(T, L^p(\Omega)) = \mathcal{E}_M(T, L^p(\Omega))/\mathcal{N}(T, L^p(\Omega)).$$

We shall consider special elements of $\mathcal{G}(T, L^p(\Omega))$ which are important for practical use.

We say that $G \in \mathcal{G}(T, L^p(\Omega))$ has a modification if there is a representative of G, R_G which belongs to $\mathcal{E}_M(\Omega, T)$, which means that R_G satisfies both conditions (*) and (**). A subspace of $\mathcal{G}(T, L^p(\Omega))$ with elements having modifications will be denoted by $\tilde{\mathcal{G}}(T, L^p(\Omega))$.

Definition 1. *Let $G \in \tilde{\mathcal{G}}(T, L^p(\Omega))$ with the modification R_G. Then the mathematical expectation of G is defined as an element of $\mathcal{G}(T)$ represented by*

$$m(\varphi, x) = E(R_G(\varphi, x)).$$

Definition 2. *Let $G \in \tilde{\mathcal{G}}(T, L^p(\Omega))$ with the modification R_G. Then, the correlation function of G is defined as an element of $\mathcal{G}(T \times T)$ represented by*

$$B_{\varphi \bigotimes \psi}(x, y) = E(R_G(\varphi, x) R_G(\psi, y)).$$

4. Derivation of C g.r.p.

In $\mathcal{G}(T, L^p(\Omega))$ we shall consider two types of derivation.

1. Almost sure derivation: If $G = [R_G] \in \mathcal{G}(T, L^p(\Omega))$ then $\partial_x G = [\partial_x R_G] \in \mathcal{G}(T, L^p(\Omega))$ is almost sure derivative of G if for every $x \in T$

$$\frac{R_G(\omega, \varphi, x+h) - R_G(\omega, \varphi, x)}{h} \to \partial_x R_G(\omega, \varphi, x), \quad h \to 0, \ a.e.\ \omega.$$

2. L^p derivation: If $G = [R_G] \in \mathcal{G}(T, L^p(\Omega))$ then $F = [R_F] \in \mathcal{G}(T, L^p(\Omega))$ is L^p derivation of G, $F = G'$ if for every

$$\frac{R_G(\omega, \varphi, x+h) - R_G(\omega, \varphi, x)}{h} \overset{L^p}{\to} R_F, \quad h \to 0.$$

We shall show that in $\tilde{\mathcal{G}}$ these two notions coincide.

Theorem 1. *Let $G \in \tilde{\mathcal{G}}(T, L^p(\Omega))$ with modification R_G. Then G' is represented by $\partial_x R_G$.*

Proof. Let $F = [R_F]$ be L^p derivative of $G = [R_G] \in \tilde{\mathcal{G}}(T, L^p(\Omega))$. It means that

$$\frac{R_G(\omega,\varphi,x+h) - R_G(\omega,\varphi,x)}{h} \xrightarrow{L^p} R_F, \quad h \to 0.$$

We have that almost sure derivation of G is represented by $\partial_x R_G$:

$$\frac{R_G(\omega,\varphi,x+h) - R_G(\omega,\varphi,x)}{h} \xrightarrow{a.e.} \partial_x R_G, \quad h \to 0.$$

However, since the limit of a sequence $X_n \in L^p$ in L^p is unique, we have $X = X_1$. (see [3]). Hence, $R_F = \partial_x R_G$.

Of course, the theorem holds for derivations of higher order:

Theorem 2. *Let $G \in \tilde{\mathcal{G}}(T, L^p(\Omega))$ with modification R_G. Then $G^{(\alpha)}$ is represented by $\partial_x^{(\alpha)} R_G$.*

The conection between the exisistence of L^2 derivatives of $G \in \tilde{\mathcal{G}}(T, L^2(\Omega))$ and derivatives of the expectation and correlation function gives the following theorem [4].

Theorem 3. *The Colombeau g.r.p. $G \in \tilde{\mathcal{G}}(T, L^2(\Omega))$ with modification R_G has L^2 derivative in $x_0 \in T$ if and only if there exist derivative $m'_G(\varphi,x) = [E(R_G(\varphi,x))]'$, at x_0 and the second derivative $\frac{\partial^2}{\partial x \partial y} B^G_{\varphi \otimes \psi}(x,y)$ at (x_0,x_0). Also,*

$$m'_G(\varphi,x) = m_{G'}(\varphi,x), \quad x \in T,$$

and

$$\frac{\partial^2}{\partial x \partial y} B^G_{\varphi \otimes \psi}(x,y) = B^{G'}_{\varphi \otimes \psi}(x,y), \quad x,y \in T.$$

Now, since G' is represented by R_G from Theorem 1 and Theorem 3 it follows:

Corollary 1. *Let $G \in \tilde{G}(T, L^2(\Omega))$ with the modification R_G Then*

$$\frac{\partial}{\partial x} m(\varphi,x) = E(\frac{\partial}{\partial x} R_G(\varphi,x)), \quad x \in T, \tag{1}$$

$$(2)\qquad \frac{\partial^2}{\partial x \partial y} B_{\varphi \bigotimes \psi}(x,y) = E[\frac{\partial}{\partial x}(R_G(\varphi,x)\frac{\partial}{\partial y}R_G(\psi,y))],\ x,y\in T.$$

Since the random variable $[R_G(\varphi,x)]^2$ is positive, its mean $B_{\varphi \bigotimes \varphi}(x,x) = E[R_G^2(\varphi,x)]$ is also positive so the correlation function $B_{\varphi \bigotimes \psi}(x,y)$ is positive definite. Moreover, the covariance function $C_{\varphi \bigotimes \psi}(x,y)$ defined by

$$C_{\varphi \bigotimes \psi}(x,y) = B_{\varphi \bigotimes \psi}(x,y) - m(\varphi,x)m(\psi,y),$$

where $m(\varphi,x) = E[R_G(\varphi,x)]$, is positive definite. Indeed,

$$C_{\varphi \bigotimes \varphi}(x,x) = B_{\varphi \bigotimes \varphi}(x,x) - m(\varphi,x)m(\varphi,x) =$$

$$= E[R_G(\varphi,x)R_G(\varphi,x)] - 2E[R_G(\varphi,x)]m(\varphi,x) + m^2(\varphi,x) =$$
$$E[|R_G(\varphi,x) - m(\varphi,x)|^2] \geq 0.$$

5. Gaussian Colombeau generalized random process

Definition 3. *Let $G \in \tilde{\mathcal{G}}(T, L^2(\Omega))$ with modification R_G. It is said that G is a Gaussian Colombeau grp (GC g.r.p.) if for any $\varphi_1,...,\varphi_n \in \mathcal{A}_0$ and $x_1,...,x_n \in \mathbf{R}$ the probability that the random variable $\mathbf{X} = (R_G(\varphi_1,x_1), ..., R_G(\varphi_n,x_n))$ belongs to a Borel set S, is*

$$(3)\qquad P\{\mathbf{X}\in S\} = \int_S \frac{\sqrt{det\mathbf{\Lambda}}}{(2\pi)^{n/2}} exp\{-\frac{1}{2}(\mathbf{\Lambda}t,t)\}dt,$$

where $\mathbf{\Lambda}$ is a non-degenerate positive definite matrix, and

$$(\mathbf{\Lambda}t,t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_{ij}t_i t_j.$$

Theorem 4. *Let G be a GC g.r.p. with modification R_G. Then for any independent functions $\varphi_1,...,\varphi_n \in \mathcal{A}_0$ and $x_1,...,x_n \in \mathbf{R}$,*

$$(4)\qquad \mathbf{\Lambda} = ||B_{\varphi_i \bigotimes \varphi_j}(x_i,x_j)||^{-1}.$$

Proof. By the definition of the correlation function of GC grp we have

$$B_{\varphi_i \bigotimes \varphi_j}(x_i, x_j) = E[R_G(\varphi_i, x_i)R_G(\varphi_j, x_j)].$$

The random variable $(R_G(\varphi_i, x_i)R_G(\varphi_j, x_j))$ if a function of $n-$dimensional random variable $\mathbf{X}$ whose distribution function is given by (3). Thus,

$$(5) \qquad E[R_G(\varphi_i, x_i)R_G(\varphi_j, x_j)] = \frac{\sqrt{det\mathbf{\Lambda}}}{(2\pi)^{n/2}} \int t_i t_j exp\{-\frac{1}{2}(\mathbf{\Lambda}t, t)\}dt.$$

Now, following [2], pp 249, we compute integral (5) using the formula

$$(6) \qquad \frac{\sqrt{detC}}{(2\pi)^{n/2}} \int (At, t)exp[-\frac{1}{2}(Ct, t)]dt = Tr(AC^{-1})$$

which is valid for any strictly positive definite matrix C and any matrix A. We have that $t_i t_j = (A_{ij}t, t)$, where A_{ij} is the matrix, whose elements all vanish with the exception of a_{ij}, which equals 1. The integral in (3) is equal to $Tr(A_{ij}\mathbf{\Lambda})$. Since $A_{ij}\mathbf{\Lambda}$ is the matrix whose all rows vanish with the exception of the ith row, which coincides with the jth row of $\mathbf{\Lambda}^{-1}$. Therefore, $tr(A_{ij}\mathbf{\Lambda}^{-1}) = \mu_{ij}$, where μ_{ij} are elements of $\mathbf{\Lambda}$, and

$$E[R_G(\varphi_i, x_i), R_G(\varphi_j x_j)] = \mu_{ij},$$

Theorem 5. *The derivatives of a Gaussian Colombeau generalized process G is again a Gaussian Colombeau generalized process.*

Proof. Since $G \in \tilde{\mathcal{G}}(T, L^p(\Omega))$, we have that its derivative G' is determined by the derivative $\partial_x R_G$ of the modification R_G, where the family $\{R_G(\omega, \varphi, x);\ (\varphi, x) \in \mathcal{A}_0 \times T\}$ is Gaussian. Thus, we have that for almost every $\omega \in \Omega$

$$\partial_x R_G(\omega, \varphi, x) = \lim_{h \to 0} \frac{R_G(\omega, \varphi, x + h) - R_G(\omega, \varphi, x)}{h}.$$

Since for the Gaussian variables an almost sure convergence implies the L^2 convergence, we can consider the linear closure of the familly $\{R_G(\omega, \varphi, x);$ $(\varphi, x) \in \mathcal{A}_0 \times T\}$ which is Gaussian. The familly $\mathbf{Z} = \{\partial_x R_G(\omega, \varphi, x);$ $(\varphi, x) \in \mathcal{A}_0 \times T\}$ is a subfamilly of $\mathbf{Z}$, and therefore Gaussain.

6. Wiener process

The space $\mathcal{CP}$ of ordinary stochastic processes with almost continuous paths can be embedded in $\tilde{\mathcal{G}}(T, L^2(\Omega))$ by

$$\mathcal{CP} \ni Z \to ((\omega, \varphi, x) \to \int_{\mathbf{R}} Z(s,\omega)\varphi_\varepsilon(s-x)ds).$$

Let W be a Wiener process. It is an element of $\mathcal{CP}$ and

$$W * \varphi_\varepsilon = \int_{\mathbf{R}} W(s)\varphi_\varepsilon(s-x)ds$$

is the corresponding element of $\tilde{\mathcal{G}}(\Omega, T, L^2(\Omega))$. Its mathematical expectation is zero, and the correlation function is given by

$$B^W_{\varphi\bigotimes\psi}(x,y) = E[(W * \check{\varphi})(x)(W * \check{\psi})(y)] =$$

$$\int_{\mathbf{R}}\int_{\mathbf{R}} E[W(t)W(s)]\varphi(t-x)\psi(s-y)dtds.$$

Since

$$E[W(t)W(s)] = \begin{cases} min\{t,s\}, & t,s \geq 0, \\ 0, & t < 0, \text{ or } s < 0 \end{cases}$$

we have, after integration by parts:

$$\int_0^\infty \int_0^\infty min\{t,s\}\varphi(t-x)\psi(s-y)dtds =$$

$$= \int_0^\infty \varphi(t-x)\int_0^t s\psi(s-y)dsdt + \int_0^\infty \psi(s-y)\int_0^s t\varphi(t-x)dtds =$$

$$= \int_0^\infty (\int_t^\infty \varphi(v-x)dv \int_t^\infty \psi(u-y)du)dt.$$

The derivative of W is determined by

$$\frac{\partial^2}{\partial x \partial y} B^W_{\varphi\bigotimes\psi}(x,y) = \int_0^\infty (\int_t^\infty \varphi'(v-x)dv \int_t^\infty \psi'(u-y)du)dt =$$

$$= \int_0^\infty \varphi(t-x)\psi(t-y)dt.$$

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