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CONTINUOUS DEPENDENCE OF THE FIXED POINTS ON PARAMETERS IN PROBABILISTIC METRIC SPACES

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Abstract

In this paper a theorem on continuous dependence of the fixed points on parameters in probabilistic metric spaces is proved.

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1. Introduction and Preliminaries

The first result from the fixed point theory in probabilistic metric spaces is obtained by Sehgal and Bharucha - Reid in [8]. Since then many fixed point theorems for singlevalued and multivalued mappings in probabilistic metric spaces are proved [3], [4], [5], [6].

In [3] a result on continuous dependence of the fixed points on parameters of certain condensing operators in probabilistic metric spaces is obtained.

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In this paper we shall prove, using the function of Kuratowski, a result on continuous dependence of the fixed points on parameters in probabilistic metric spaces.

An application of the theorem is also given.

By Δ we shall denote the set of all distribution functions F such that $F(0)=0)$ (F is a nondecreasing, leftcontinuous mapping from $\mathbf{R}$ into $[0,1]$ so that $\sup\limits_{x\in\mathbf{R}} F(x)=1$).

The ordered pair $(S,\mathcal{F})$ is a probabilistic metric space [7] if S is a nonempty set and $\mathcal{F}: S\times S\to\Delta$ a symmetric function ($\mathcal{F}(p,q)$ is denoted by $F_{p,q}$, for every $(p,q)\in S\times S$) satisfies the following conditions:

1. $F_{u,v}(x)=1$, for every $x>0 \Rightarrow u=v$ $(u,v\in S)$.

2. $F_{u,v}(x)=1$ and $F_{v,w}(y)=1 \Rightarrow F_{u,w}(x+y)=1$ for $u,v,w\in S$ and $x,y\in\mathbf{R}^+$.

A Menger space is a triple $(S,\mathcal{F},T)$, where $(S,\mathcal{F})$ is a probabilistic metric space and T is a $t-$ norm [7].

The $(\epsilon,\lambda)-$ topology in S is introduced by the family of neighbourhoods given by

$$\mathcal{U}=\{U_v(\epsilon,\lambda)\}_{(v,\epsilon,\lambda)\in S\times R^+\times(0,1)}, \quad \text{where}$$

$$U_v(\epsilon,\lambda)=\{u;\ F_{u,v}(\epsilon)>1-\lambda\}.$$

It $t-$ norm T is continuous then S is, in the (ϵ,λ) topology, a metrizable topological space.

Let $(S,\mathcal{F})$ be a probabilistic metric space. In [1] the notions of probabilistic diameter and the Kuratowski function are given.

Definition 1. *Let A be a nonempty subset of S. The function $D_A(\cdot)$, defined in R^+ by*

$$D_A(u)=\sup_{s<u}\inf_{p,q\in A} F_{p,q}(s),\quad u\in R^+,$$

is called the probabilistic diameter of the set A and the set A is probabilistic bounded if and only if

$$\sup_{u\in R^+} D_A(u)=1.$$

Definition 2. *Let A be a probabilistic bounded subset of S. The Kuratowski function $\alpha_A : R^+ \to [0,1]$ is defined by $\alpha_A(u) = \sup\{\epsilon;\ \epsilon > 0$, there is a finite family $\{A_j\}_{j\in J}$ in S such that $A = \bigcup_{j\in J} A_j$ and $D_{A_j}(u) \geq \epsilon$, for every $j \in J\}$.*

The Kuratowski function has the following properties:

1) $\alpha_A \in \Delta$.

2) $\alpha_A(u) \geq D_A(u)$, for every $u \in R^+$.

3) $\emptyset \neq A \subset B \subset S \Rightarrow \alpha_A(u) \geq \alpha_B(u)$, for every $u \in R^+$.

4) $\alpha_{A\cup B}(u) = \min\{\alpha_A(u), \alpha_B(u)\}$, for every $u \in R^+$.

5) $\alpha_A(u) = \alpha_{\overline{A}}(u)(u \in R^+)$, where $\overline{A}$ is the closure of A.

6) $\alpha_A = H \iff A$ is precompact, where

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

The function $\beta_A : R^+ \to [0,1]$ is defined by

$\beta_A(u) = \sup\{\epsilon;\ \epsilon > 0$, there exists a finite subset A_f of S such that $\tilde{F}_{A,A_f}(u) \geq \epsilon\}$ where

$$\tilde{F}_{A,B}(u) = \sup_{s<u} \inf_{x\in A} \sup_{y\in B} F_{x,y}(s),$$

for probabilistic bounded subsets $A, B \subset S$.

Let $(S, \mathcal{F})$ be a probabilistic metric space, K a probabilistic bounded subset of S and $A : K \to 2^S \setminus \emptyset$. If $A(K)$ is probabilistic bounded subset of S and for every $B \subset K$:

$\gamma_{A(B)}(u) \leq \gamma_B(u)$, for every $u > 0 \Rightarrow B$ is precompact,
where γ_B is α_B or β_B for $B \subset S$ then we say that A is densifying on the set K in respect to the function γ.

2. A theorem on continuous dependence on parameters

In the next Lemma Φ is the set of all strictly monotone increasing functions $g : [0, \infty) \to [0, \infty)$.

Lemma 1. *Let (S, F, t) be a Menger space with a continuous $T-$ norm t, M a nonempty closed and probabilistic bounded subsets of S, Λ a metric space and $G : M \times \Lambda \to M$ so that the following conditions are satisfied:*

(i) There exists $g \in \Phi$ so that for every $B \subseteq M$, every $u > 0$ and every $\lambda \in \Lambda$:

$$\alpha_{G(B,\lambda)}(g(u)) \geq \alpha_B(u).$$

(ii) The mapping $\lambda \mapsto G(x, \lambda)$ $(\lambda \in \Lambda)$ is continuous, uniformly in respect to $x \in M$.

Then for every compact subset Λ_0 of Λ and every $B \subseteq M$:

$$\alpha_{G(B,\Lambda_0)}(g(u)) \geq \alpha_B(u), \quad \text{for every} \;\; u > 0. \tag{1}$$

Proof. In order to prove (1) we shall prove that for every $u > 0$, every $s \in (0, u)$ and every $B \subseteq M$:

$$\alpha_{G(B,\Lambda_0)}(g(u)) \geq \alpha_B(u - s). \tag{2}$$

Since the mapping $\alpha_B(\cdot)(B \subseteq M)$ is left continuous from (2) we shall obtain that:

$$\lim_{s \to 0} \alpha_B(u - s) = \alpha_B(u) \leq \alpha_{G(B,\Lambda_0)}(g(u)), \;\; u > 0.$$

We shall prove the following implication:

$$0 < r < \alpha_B(u - s) \Rightarrow r \leq \alpha_{G(B,\Lambda_0)}(g(u)).$$

Suppose that $0 < r < \alpha_B(u - s)$ and prove that for every $h \in (0, r)$

$$r - h \leq \alpha_{G(B,\Lambda_0)}(g(u)).$$

The mapping $(u,v) \mapsto (u,t(r,v))$ is continuous and since $t(1,t(r,1)) = r$ it follows that there exists $\bar{h} \in (0,1)$ so that:

$$1 \geq u,v > 1-\bar{h} \Rightarrow t(u,t(r,v)) > r-h.$$

Since the mapping $\lambda \mapsto G(x,\lambda)$ is continuous, uniformly in respect to x, there exists for every $\bar{\lambda} \in \Lambda_0$, a $\rho(\bar{\lambda}) > 0$ so that:

$$d(\lambda,\bar{\lambda}) < \rho(\bar{\lambda}) \Rightarrow F_{G(x,\lambda),G(x,\bar{\lambda})}\Big(\frac{g(u)-g(u-s)}{8}\Big) > \eta, \text{ for every } x \in M$$

where $t(\eta,\eta) > 1-\bar{h}$. Then $\Lambda_0 \subseteq \bigcup_{\lambda \in \Lambda_0} L(\lambda, \frac{\rho(\lambda)}{2})(L(a,v)$ is the ball with the centre $a \in \Lambda$ and the radious $v)$ and since Λ_0 is a compact set there exists $\{\lambda_1,\lambda_2,...,\lambda_k) \subseteq \Lambda_0$ so that:

$$\Lambda_0 \subseteq \bigcup_{i=1}^{k} L\Big(\lambda_i, \frac{\rho(\lambda_i)}{2}\Big). \tag{3}$$

We shall prove the following implication:

$$x \in M,\ d(\lambda',\lambda'') < \rho \Rightarrow F_{G(x,\lambda'),G(x,\lambda'')}\frac{(g(u)-g(u-s)}{4}) > 1-\bar{h}$$

where

$$\rho = \min_{1 \leq i \leq k}\{\rho(\lambda_i)2^{-1}\}.$$

Suppose that $d(\lambda',\lambda'') < \rho$.

From relation (3) it follows that there exists $\lambda_i \in \Lambda_0$ so that $d(\lambda',\lambda_i) < \rho(\lambda_i)2^{-1}$. Then we have that:

$$d(\lambda'',\lambda_i) \leq d(\lambda'',\lambda') + d(\lambda',\lambda_i) \leq \rho(\lambda_i)$$

which implies that:

$$F_{G(x,\lambda'),G(x,\lambda'')}\Big(\frac{g(u)-g(u-s)}{4}\Big)$$

$$\geq t(F_{G(x,\lambda'),G(x,\lambda_i)}\Big(\frac{g(u)-g(u-s)}{8}\Big), F_{G(x,\lambda_i),G(x,\lambda'')}\Big(\frac{g(u)-g(u-s)}{8}\Big)\Big) \geq$$

$$\geq t(\eta,\eta) > 1 - \bar{h}.$$

Let $\Lambda_0 = \bigcup_{i=1}^{n} S_i$ such that diam $S_i < \rho$ and $\bar{\lambda}_i \in S_i$ for every $i \in \{1,2,...,n\}$. First, we shall prove that for every $i \in \{1,2,...,n\}$ and $h \in (0,r)$:

$$\alpha_{G(B,S_i)}(g(u)) \geq r - h. \tag{4}$$

In order to prove (4) we shall prove that there exists a finite family $\{A_j\}_{j=1}^{l(i)}$ such that:

$$G(B,S_i) \subseteq \bigcup_{j=1}^{l(i)} A_j \text{ and } \alpha_{A_j}(g(u)) \geq r - h,\ j \in \{1,2,...,l(i)\}. \tag{5}$$

Then from (5) we shall obtain that:

$$\alpha_{G(B,S_i)}(g(u)) \geq \alpha_{\bigcup_{j=1}^{l(i)} A_j}(g(u)) = \min_{1\leq j\leq l(i)} \alpha_{A_j}(g(u)) \geq r - h.$$

Since $0 < r < \alpha_B(u-s) \leq \alpha_{G(B,\bar{\lambda}_i)}(g(u-s))$ from the definition of the function $\alpha(\cdot)$ it follows that there exists a finite family $\{B_1, B_2, ..., B_{l(i)}\}$ in S_i such that:

$$G(B,\bar{\lambda}_i) = \bigcup_{j=1}^{l(i)} B_j,\ D_{B_j}(g(u-s)) > r,\ j \in \{1,2,...,l(i)\} \tag{6}$$

Hence

$$\sup_{c<g(u-s)} \inf_{v,w\in B_j} F_{v,w}(c) > r$$

which implies that $F_{v,w}(g(u-s)) > r$, for every $v,w \in B_j$ and for every $j \in \{1,2,...,l(i)\}$. Let $A_j = B_j \cup C_j$, $j \in \{1,2,...,l(i)\}$ where:

$C_j = \{x | x \in G(B,S_i)$, there exists $z \in B_j$ such that

$$F_{z,x}\Big(\frac{g(u)-g(u-s)}{4}\Big) > 1 - \bar{h}\}.$$

We shall prove that (5) is satisfied. If $x \in G(B, S_i)$ then there exist $y \in B$ and $\lambda \in S_i$ so that $x = G(y, \lambda)$. Since $(\lambda, \bar{\lambda}_i) \in S_i \times S_i$ we have that $d(\lambda, \bar{\lambda}_i) < \rho$ and

$$(7) \qquad F_{G(y,\lambda),G(y,\bar{\lambda}_i)}\Big(\frac{g(u) - g(u-s)}{4}\Big) > 1 - \bar{h}.$$

From (6) it follows that $z = G(y, \bar{\lambda}_i) \in B_j$, for some $j \in \{1, 2, ..., l(i)\}$ and so (7) implies that $G(y, \lambda) \in C_j$. This means that $x = G(y, \lambda) \in A_j$. Hence (5) will be proved if we prove that for every $j \in \{1, 2, ..., l(i)\}$:

$$(8) \qquad D_{A_j}(g(u)) = \sup_{c<g(u)} \inf_{v,w \in A_j} F_{v,w}(c) \geq r - h.$$

In order to prove (8) we shall prove that for every $v, w \in A_j$

$$F_{v,w}(2^{-1}(g(u) + g(u-s))) \geq r - h.$$

Since the mapping g is strictly monotone increasing we have that $g(u) > 2^{-1}(g(u) + g(u-s))$ and we will obtain (8). We have the following three cases:

1) Let $v, w \in B_j$. Since $2^{-1}(g(u) + g(u-s)) > g(u-s)$ we have that (6) implies:

$$F_{v,w}(2^{-1}(g(u) + g(u-s))) \geq F_{v,w}(g(u-s)) > r > r - h.$$

2) Let $v \in B_j$ and $w \in C_j$. Then there exists $z \in B_j$ so that $F_{w,z}\Big(\frac{g(u) - g(u-s)}{4}\Big) > 1 - \bar{h}$ which implies that:

$$F_{v,w}(2^{-1}(g(u) + g(u-s))) \geq t(F_{v,z}(g(u-s)),$$

$$F_{z,w}(2^{-1}(g(u) - g(u-s)) \geq t(F_{v,z}(g(u-s)),$$

$$t(1, F_{z,w}(4^{-1}(g(u) - g(u-s))))$$

$$\geq t(1, t(r, F_{z,w}(4^{-1}(g(u) - g(u-s)))) > r - h.$$

3) Let $v, w \in C_j$. Then there exist $\tilde{v} \in B_j$ and $\tilde{w} \in B_j$ so that

$$F_{\tilde{v},v}(4^{-1}(g(u) - g(u-s))) > 1 - \bar{h},$$

$$F_{\tilde{w},w}(4^{-1}(g(u)-g(u-s)))>1-\bar{h}.$$

Then we have that:

$$F_{v,w}(2^{-1}(g(u)+g(u-s)))\geq t(F_{\tilde{v},v}(4^{-1}(g(u)-g(u-s))),$$
$$t(F_{\tilde{v},\tilde{w}}(g(u-s)),F_{\tilde{w},w}(4^{-1}(g(u)-g(u-s)))))>r-h.$$

This implies (8). Since $\alpha_{A_j}(g(u))\geq D_{A_j}(g(u))$ we obtain that for every $j\in\{1,2,...,l(i)\}$, $\alpha_{A_j}(g(u))\geq r-h$ and so $\alpha_{G(B,S_i)}(g(u))\geq r-h$, for every $i\in\{1,2,...,n\}$. It is obvious that this implies:

$$\alpha_{G(B,\Lambda_0)}(g(u))=\alpha_{G(B,\bigcup_{i=1}^{n}S_i)}(g(u))=$$
$$=\min_{1\leq i\leq n}\alpha_{G(B,S_i)}(g(u))\geq r-h.$$

Since h is an arbitrary number from $(0,r)$ we obtain that $\alpha_{G(B,\Lambda_0)}(g(u))\geq r$.

Using the lemma we shall prove the following theorem on continuous dependence of the fixed points on parameters.

Theorem 1. *Let (S,F,t) be a complete Menger space with continuous $T-$norm t,M a nonempty closed and probabilistic bounded subset of S,Λ a complete metric space and $G:M\times\Lambda\to M$ so that the following conditions are satisfied:*

(i) For every $\lambda\in\Lambda$, the mapping $x\mapsto G(x,\lambda)$ is continuous $(x\in M)$ and for every $x\in M$ the mapping $\lambda\mapsto G(x,\lambda)$ is continuous $(\lambda\in\Lambda)$, uniformly with respect to $x\in M$.

(ii) For each $\lambda\in\Lambda$, the equation $x=G(x,\lambda)$ has a solution in M.

(iii) For every $B\subseteq M$, every $u>0$ and every $\lambda\in\Lambda$:

$$\alpha_{G(B,\lambda)}(g(u))\geq\alpha_B(u)$$

where $g\in\Phi$ and $\lim_{n\to\infty}(g^{-1})^n u=\infty$, for every $u>0$.

Then $F(\lambda)$ is upper semicontinuous at each $\lambda\in\Lambda$, where:

$$F(\lambda)=\{x|x\in M,\ \ x=G(x,\lambda)\}.$$

Proof. First, we shall prove that for every compact set $\Lambda_0 \subseteq \Lambda$ and $B \subseteq M$:

$\alpha_B \neq H \Rightarrow$ there exists $u_0 > 0$ such that $\alpha_{G(B,\Lambda_0)}(u_0) > \alpha_B(u_0)$.

If we suppose that $\alpha_B \neq H$ and $\alpha_{G(B,\Lambda_0)}(u) \leq \alpha_B(u)$, for every $u > 0$ then from the lemma we have that:

$$\alpha_B(u) \leq \alpha_{G(B,\Lambda_0)}(g(u)) \leq \alpha_B(g(u)),$$

for every $u > 0$ and so:

$$\alpha_B(u) \geq \alpha_B(g^{-1}(u)), \quad \text{for every } u > 0.$$

Since $\alpha_B(\cdot)$ is such a function that $\lim_{u\to\infty} \alpha_B(u) = 1$, from the condition $\lim_{n\to\infty} (g^{-1})^n(u) = \infty$ it follows that for every $u > 0$, $\alpha_B(u) = 1$, which means that $\alpha_B = H$.

This is a contradiction and so the above implication is proved. Now, we can prove that the mapping F is upper semicontinuous. Suppose, on the contrary, that F is not an upper semicontinuous mapping at some $\lambda_0 \in \Lambda$. Then there exists an open set $O \supset F(\lambda_0)$ such that for every $\delta > 0$ there exists $\lambda(\delta) \in \Lambda$ so that $d(\lambda(\delta), \lambda_0) < \delta$ and $F(\lambda(\delta)) \not\subset O$. As in [3] let $\delta_1 > \delta_2 > \dots$, $\lim_{n\to\infty} \delta_n = 0$. Then $\lambda(\delta_n) = \lambda_n \to \lambda_0$ and let $x_n \in F(\lambda_n) \setminus O$, for every $n \in N$. We shall prove that there is a convergent subsequence of the sequence $\{x_n\}_{n\in \mathrm{N}}$. Let $\Lambda_0 = \{\lambda_n | n \in N\}$ and $B = \{x_n | n \in N\}$. Suppose that $\alpha_B < H$. For every $n \in N$ we have that $x_n = G(x_n, \lambda_n)$ and so for every $u > 0$:

$$\alpha_B(u) = \alpha_{\{G(x_n,\lambda_n)|n\in N\}}(u) \geq \alpha_{G(B,\Lambda_0)}(u)$$

$\geq \alpha_{G(B,\bar{\Lambda}_0)}(u)$. Since there exists $u_0 > 0$ such that

$$\alpha_{G(B,\bar{\Lambda}_0)}(u_0) > \alpha_B(u_0)$$

we obtain that $\alpha_B(u_0) > \alpha_B(u_0)$ which is a contradiction. Hence $\alpha_B = H$, which implies that B is a relatively compact set. Hence, there exists a convergent subsequence $\{x_{n_k}\}_{k\in N}$. Suppose that $\lim_{k\to\infty} x_{n_k} = x_0 \in M$. Then from $\lim_{k\to\infty} \lambda_{n_k} = \lambda_0$ we obtain that $x_{n_k} = G(x_{n_k}, \lambda_{n_k})$ tends to $G(x_0, \lambda_0)$. This implies that $x_0 = G(x_0, \lambda_0)$ and so $x_0 \in F(\lambda_0)$. On the other hand x_n belongs, for every $n \in N$, to the complement of O which implies that $x_0 \notin O \supset F(\lambda_0)$.

Hence, we obtain a contradiction which means that F is an upper semicontinuous mapping.

Proposition 1 *Let (S, F, t) be a complete Menger space with a continuous $T-$ norm t, M a nonempty closed and probabilistic bounded subset of $S, \tilde{\Lambda}$ a complete metric space, $Q : M \to \tilde{\Lambda}$ a compact mapping (Q is continuous and $\overline{Q(M)}$ is compact) and $G : M \times \overline{Q(M)} \to M$ such that all the conditions of the lemma are satisfied for $\Lambda = \overline{Q(M)}$. Then the mapping $x \mapsto G(x, Qx)$ $(x \in M)$ is a probabilistic $g-$ condensing mapping.*

Proof. We have to prove that for every subset $B \subseteq M$ and every $u > 0$: $\alpha_{\{G(x,Qx)|x\in B\}}(g(u)) \geq \alpha_B(u)$.

From the lemma it follows that

$$\alpha_{G(B,\overline{Q(M)})}(g(u)) \geq \alpha_B(u)$$

and since

$$\alpha_{\{G(x,Qx)|x\in B\}}(u) \geq \alpha_{G(B,\overline{Q(M)})}(u) \quad (u > 0)$$

we obtain that:

$$\alpha_{\{G(x,Qx)|x\in B\}}(g(u)) \geq \alpha_{G(B,\overline{Q(M)})}(g(u)) \geq \alpha_B(u)$$

for every $u > 0$.

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