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INTEGRAL WITH RESPECT TO FUZZY MEASURE

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Abstract

Fuzzy measure is the mapping which maps σ-algebra $\mathcal{A}$ to the set of all fuzzy numbers. The integral with respect to fuzzy measure is defined and some properties are investigated.

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1. Introduction

In the last decade the subject of integration with respect to a fuzzy measure has been investigated extensively in connection with various problems in fuzzy random variable.

The concept of fuzzy measure has been formalized in two direction: fuzzy measure as nonadditive, monotone function of a set with values in R and fuzzy measure as an additive function which maps the subset to a fuzzy set. This second concept of fuzzy measure is a natural generalization of set valued measure. Contributions in this field were made, among others, by Puri, Ralescu [8], Ban [1], Stojaković [10].

Using an additive fuzzy measure a notion of integral of single valued function is defined and some basic properties are investigated.

2. Additive fuzzy measure

Throughout this paper let $\mathbf{R}$ be the set of reals. A fuzzy set $u \in \mathcal{F}$ is a function $u : \mathbf{R} \to [0,1]$ for which the α-level set u_α of u, defined by

$$u_\alpha = \{x \in \mathbf{R} : u(x) \geq \alpha\},\ u_0 = \{x \in R : u(x) > 0\}$$

is nonempty, compact and convex subset of $\mathbf{R}$ for all $\alpha \in [0,1]$. We call them fuzzy numbers.

Let $(\Omega, \mathcal{A})$ be a measureable space with $\mathcal{A}$ a σ-field of measurable subsets of the set Ω. The mapping $\mathcal{M} : \mathcal{A} \to \mathcal{F}$ such that for every sequence $\{A_i\}_{i\in N}$ of pairwise disjoint elements of $\mathcal{A}$ such that $\sum_{i=1}^{\infty} |\mathcal{M}_\alpha(A_i)| < \infty$ for all $\alpha \in (0,1]$, the following equality is satisfied

$$\mathcal{M}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathcal{M}(A_i)$$

where

$$(\sum_{i=1}^{n} \mathcal{M}(A_i))(x) = \sup\{\bigwedge_{i=1}^{\infty} \mathcal{M}(A_i)(x_i) \ : \ x = \sum_{i=1}^{\infty} x_i\}$$

we shall call a fuzzy measure.

If $\mathcal{M}$ is a fuzzy measure, then for every $\alpha \in (0,1], \mathcal{M}_\alpha : \mathcal{A} \to 2^{\mathbf{R}}$ with compact convex images defined by $\mathcal{M}_\alpha(A) = [\mathcal{M}(A)]_\alpha, A \in \mathcal{A}$, is a set-valued measure. Conversely, if $\{\mathcal{M}_\alpha\}_{\alpha\in(0,1]}$ is a family of compact, convex set valued measures such that $\mathcal{M}_\alpha = \bigcap_{i=1}^{\infty} \mathcal{M}_{\alpha_i}$ for every nondecreasing sequence $\{\alpha_i\} \subset [0,1]$, $\lim_{i\to\infty} \alpha_i = \alpha$, then $\mathcal{M} : \mathcal{A} \to \mathcal{F}$ defined by $\mathcal{M}(A)(x) = \sup\{\alpha : x \in \mathcal{M}_\alpha(A)\}$ is a fuzzy valued measure.

We say that a fuzzy valued measure $\mathcal{M}$ is μ-continuous, where $\mu : \mathcal{A} \to \mathbf{R}$ is a single valued measure, if $\mu(A) = 0$ implies $\mathcal{M}(A) = I_{\{0\}}$. Further, $\mathcal{M}$ is of bounded variation if

$$\sup \sum_i \sup_{x\in\mathcal{M}_0(A_i)} \|x\| < \infty$$

where the suppremum is taken over all finite measurable partition of Ω.

3. Integration

Let $(\Omega, \mathcal{A}, \mu)$ be a σ–finite measure space, $\mathcal{M} : \mathcal{A} \to \mathcal{F}$ be a fuzzy measure, $S_{\mathcal{M}_\alpha}$ the set of all measure selection of $\mathcal{M}_\alpha$ and $f : \Omega \to \mathbf{R}$ be an integrable function (with respect to μ), then for every $\alpha \in (0, 1]$ we can define

$$\phi_\alpha(A) \stackrel{\text{def}}{=} \int_A f d\mathcal{M}_\alpha \stackrel{\text{def}}{=} \{\int_A f dm, \;\; m \in S_{\mathcal{M}_\alpha}\}.$$

Theorem 1. *Let $\mathcal{M} : \mathcal{A} \to \mathcal{F}$ is a μ-continuous fuzzy measure of bounded variation and $f : \Omega \to \mathbf{R}$ be an integrable function. Then, for every $A \in \mathcal{A}$ the mapping $\phi(A) : \mathbf{R} \to [0, 1]$ defined by*

$$\phi(A)(x) = \sup\{\alpha \in (0, 1] : x \in \phi_\alpha(A)\}$$

is one and only one fuzzy number.

Proof. Since $\mathcal{M}$ satisfies all the conditions of Th. 4 [10], there exists an integrably bounded fuzzy valued function $X : \Omega \to \mathcal{F}$ which is the Radon-Nikodým derivative of $\mathcal{M}$. Then

$$\mathcal{M}_\alpha(A) = \int_A X_\alpha d\mu$$

for all $A \in \mathcal{A}$ and all $\alpha \in (0, 1]$ which means that $X_\alpha = \frac{d\mathcal{M}_\alpha}{d\mu}$.

In order to prove that

$$\int_\Omega f d\mathcal{M}_\alpha = \int_\Omega f X_\alpha d\mu$$

we shall show that $\int_\Omega f d\mathcal{M}_\alpha \subseteq \int_\Omega f X_\alpha d\mu$ and $\int_\Omega f X_\alpha d\mu \subseteq \int_\Omega f d\mathcal{M}_\alpha$.

If we suppose that $x \in \int_\Omega f d\mathcal{M}_\alpha$, then there exists s measure selection m_α of $\mathcal{M}_\alpha$ such that $x = \int_\Omega f dm_\alpha$. But m_α is a measure selection of $\mathcal{M}_\alpha$, which means that there exists an integrable selection h_α of X_α which is the Radon Nikodým derivative of m_α, i.e. $h_\alpha = \frac{dm_\alpha}{d\mu}$. Then

$$x = \int_\Omega f h_\alpha d\mu \in \int_\Omega f X_\alpha d\mu, \text{ that is } \int_\Omega f d\mathcal{M}_\alpha \subseteq \int_\Omega f X_\alpha d\mu.$$

On the other hand, if $x \in \int_\Omega f X_\alpha d\mu$, then there exists a measurable selection $h_\alpha \in S_{X_\alpha}$ such that $x = \int_\Omega f h_\alpha d\mu$. Since h_α is integrably bounded it follows that

$$\int_A h_\alpha d\mu = m_\alpha(A), \quad A \in \mathcal{A}$$

that is $h_\alpha = \frac{dm_\alpha}{d\mu}$. It is obvious that $m_\alpha \in S_{\mathcal{M}_\alpha}$ and

$$x = \int_\Omega f h_\alpha d\mu = \int_\Omega f dm_\alpha \in \int_\Omega f d\mathcal{M}_\alpha.$$

Having prove that $\int_\Omega f d\mathcal{M}_\alpha = \int_\Omega f X_\alpha d\mu$, it remains to recall that

$$fX_\alpha = (fX)_\alpha \text{ and } \int_\Omega f d\mathcal{M}_\alpha = \left(\int_\Omega f X d\mu\right)_\alpha.$$

The family of sets $\{\int_\Omega f d\mathcal{M}_\alpha\}_{\alpha\in(0,1]}$ generates one and only one fuzzy number $\phi(\Omega) : \mathbf{R} \to [0,1]$ defined by

$$\phi(\Omega)(x) = \sup\left\{\alpha \in (0,1] : x \in \int_\Omega f d\mathcal{M}_\alpha\right\}.$$

The preceding arguments are now repeated for any $A \in \mathcal{A}$ instead of Ω.
□

We shall write $\phi(A) = \int_A f d\mathcal{M}$. Now we list some properties of the integral with respect to fuzzy valued measure

Lemma 1. *If $f, g : \Omega \to \mathbf{R}$ are integrable functions, $\mathcal{M}$ is μ-continuous fuzzy valued measure of bounded variation and $\alpha \in \mathbf{R}$, then*

$$\int_\Omega (f+g) d\mathcal{M} = \int_\Omega f d\mathcal{M} + \int_\Omega g d\mathcal{M}$$

and

$$\int_\Omega \alpha f d\mathcal{M} = \alpha \int_\Omega f d\mathcal{M}.$$

Proof. Using results of Th.1 and Sec.3 in [11] it is easy to prove that the integral with respect to fuzzy valued measure is a linear function.

Lemma 2. *If $f : \Omega \to \mathbf{R}$ is an integrable function and $\mathcal{M} : \mathcal{A} \to \mathcal{F}$ is μ-continuous fuzzy valued measure of bounded variation, then $\phi : \mathcal{A} \to \mathcal{F}$ defined by*

$$\phi(A) = \int_A f d\mathcal{M}$$

is a fuzzy valued measure of bounded variation.

Proof. Since $\phi(A) = \int_A f d\mathcal{M} = \int_A f X d\mu$, where $X = \frac{d\mathcal{M}}{d\mu}$, from Th.3 [10] ϕ is a fuzzy valued measure of bounded variation.

The proofs for the following two lemmas are simple, so they are omitted.

Lemma 3. *Let $\mathcal{M}'$ and $\mathcal{M}''$ be μ-continuous fuzzy valued measures of bounded variation such that $\mathcal{M}'(A)(x) \leq \mathcal{M}''(A)(x)$ all $A \in \mathcal{A}$ and all $x \in \mathbf{R}$. Then*

$$\left(\int_\Omega f d\mathcal{M}'\right)(x) \leq \left(\int_\Omega f d\mathcal{M}''\right)(x) \quad \text{for all } x \in \mathbf{R}.$$

Lemma 4. *Let $(\Omega, \mathcal{A}_1)$ and $(\Omega, \mathcal{A}_2)$ be two measurable spaces, $S : \Omega \to \Omega$ be $\mathcal{A}_1$-measurable function and $f : \Omega \to R$ a bounded function. If $\mathcal{M} : \mathcal{A}_2 \to \mathcal{F}$ is a μ-continuous fuzzy valued measure of bounded variation, then for every $A \in \mathcal{A}_2$*

$$\int_{S^{-1}(A)} (f \cdot S) d\mathcal{M} = \int_A f d(\mathcal{M} \cdot S).$$

Lemma 5. *Let $\{\mathcal{A}_k\}_{k \in N}$ be an increasing sequence of sub-σ-fields of $\mathcal{A}$ such that $\bigcup_{k=1}^{\infty} \mathcal{A}_k = \mathcal{A}$ and let $\mathcal{M} : \mathcal{A} \to \mathcal{F}$ be μ-continuous fuzzy valued measure of bounded variation. If $X_n = \frac{d\mathcal{M}}{d\mu}|_{\mathcal{A}_k}$ (X_n is the Radon Nikodým derivative of $\mathcal{M}$ with respect to the restriction of μ on $\mathcal{A}_n$), then $\lim_{n \to \infty} (X_n)_\alpha \overset{\triangle}{=} X_\alpha = \left(\frac{d\mathcal{M}}{d\mu}\right)_\alpha$.*

If that convergence is uniform for $\alpha \in (0,1]$ then $\lim_{n \to \infty} X_n \overset{\mathcal{D}}{=} X$.

Proof. By Th.4 [10] $\mathcal{M}$ has a Radon-Nikodým derivative $\frac{d\mathcal{M}}{d\mu} = X$ with compact convex α-cuts. It means that

$$\mathcal{M}(A) = \int_A X d\mu \quad \text{for every } A \in \mathcal{A}.$$

Following the results of [10], [11] for every $A \in \mathcal{A}_n$

$$\int_A X d\mu = \int_A E(X|\mathcal{A}_n) d\mu$$

where $E(X|\mathcal{A}_n)$ denotes the conditional expectation of X with respect to the sub-σ-algebra $\mathcal{A}_n$. Thus, from the uniqness of the Radon-Nikodým derivative, we get

$$X_n = E(X|\mathcal{A}_n) = \frac{d\mathcal{M}}{d\mu}|_{\mathcal{A}_n}.$$

Since $(X_n)_\alpha = E(X_\alpha|\mathcal{A}_n)$, from Th.6.1 [2]

$$\lim_{n\to\infty} (X_n)_\alpha \overset{\triangle}{=} X_\alpha.$$

Now, we can apply Th.9 [11] which imply the convergence

$$\lim_{n\to\infty} X_n \overset{\mathcal{D}}{=} X.$$

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